EE341 – Discrete Time Linear Systems

1. A second course in signal processing.
2. Will study 3 new transforms: Z-transform, DTFT, DFT
3. Will use Matlab for signal processing applications
4. Will discuss filters and will filter signals in Matlab
5. Will learn Discrete Fourier Transform (DFT) which is used in many practical applications such as spectrum analyzer, EKG systems, compression systems, etc.
6. We will cover an introduction to the important area of DIGITAL SIGNAL PROCESSING.

Definition: A discrete time signal is one that is defined only for discrete points in time (hourly, every second, etc.)

Ex.: An image on a computer is a discrete signal. It is defined only at discrete points in space, called pixels.

Ex.: Any signal on a computer which is a list of numbers is a discrete signal.

Ex.: A picture taken with a digital camera is a discrete signal.

Ex.: A DVD format movie is a discrete signal.

Ex.: An MP3 file is a discrete signal.

Chapter 9 – Discrete-Time Signals and Systems

We assume that we derived a discrete-time signal from a continuous time signal via *sampling.* Given $f(t)$ to be a continuous time signal, $f(nT)$ is the value of $f(t)$ at $t = nT$. The discrete-time signal $f[n]$ is defined only for n an integer. So if we derive $f[n]$ from $f(t)$ by sampling every T seconds, where T is the sample period, we get:

$$f(nT) = f(t)|_{t=nT}$$

$$f[n] = f(nT) = f(t)|_{t=nT}$$

We will not necessarily assume that $f[n]$ is a *discrete amplitude* signal. A signal that is both discrete time and discrete amplitude is known as a *digital* signal. You will see these in later communications courses but a well-known example of a digital signal is music on a compact disk.

Note that a discrete-time signal need not be generated by explicitly sampling a continuous-time signal. Some signals are inherently discrete time, such as computer bit sequences, and some signals are implicitly sampled, such as the daily DJIA or yearly temperature averages.

9.1 Discrete-Time Signals and Systems

Remember the square brackets!

Read (skim) the example of Euler integration in the book but we will cover difference equations in Chapter 10 and when we discuss the Z-transform.

Euler integration approximates the area under a curve $x(t)$ by the sum of rectangular areas.

Discrete-Time Unit Step Function

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Notice that here, the unit step is defined at $n = 0$, unlike for continuous time.

The time-shifted unit step function $u[n - n_0]$ is:

$$u[n - n_0] = \begin{cases} 1, & n \geq n_0 \\ 0, & n < n_0 \end{cases}$$

Discrete-Time Unit Impulse Function

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Here, there is no difficulty in defining the impulse as we had in continuous time.

Shifted unit impulse:

$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

The summation is the discrete-time analog of the running integral in continuous time, and the first difference is the analog of the derivative. With these analogies, the unit impulse has essentially the same behavior in discrete and continuous time, including the sifting property.

Continuous time	Discrete time
$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$	$u[n] = \sum_{k=-\infty}^{n} \delta[k]$
$\delta(t) \equiv \frac{d}{dt} u(t)$	$\delta[n] = u[n] - u[n-1]$
$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$	$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
$\int_{-\infty}^{\infty} x(t)\delta(t-t_0) dt = x(t_0)$	$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$

Recall that continuous-time signals could be represented by an equation (which might be defined in regions) or a graph. Discrete-time signals can be represented in these ways, but also using a table. For example:

$$x[n] = u[n] - u[n-4]$$

n	≤ -1	0	1	2	3	≥ 4
$x[n]$	0	1	1	1	1	0

9.2 Transformations of Discrete-Time Signals

Note: The book uses the notation $x_t[n]$ where t is for "transformed". Since you will more typically see things like $y[n]$ and $z[n]$, I will use different notation in these notes.

Time reversal

$$y[n] = x[m]|_{m=-n} = x[-n]$$

Ex: Reading an array of numbers backwards. Taking a digital image and looking at it upside down and reversed. Playing a CD backwards.

This flips a signal about the vertical axis.

Time Scaling

$y[n] = x[m]|_{m=an} = x[an]$ SPEED UP ($|a| > 1$) or SLOW DOWN ($|a| < 1$) by a factor of a

Unlike continuous time, there are **restrictions** on a!

For **speeding up** (also known as "subsampling"), a must be an integer.
Example: For $a = 2$, you only take every other sample of $x[n]$.

Find $w_1[n] = x[2n]$ and $w_2[n] = x[2n+1]$.

For **slowing down** (expanding) a signal, you need $a = 1/K$ where K is an integer.

Example: Let $K = 2$ ($a = 1/2$) and find $z[n] = b[\frac{n}{2}]$

n	$z[n]$	$b[\frac{n}{2}]$
0	$z[0]$	$b[0]$
1	$z[1]$	??
2	$z[2]$	$b[1]$
3	$z[3]$	??

Values like $b[\frac{1}{2}]$ and $b[\frac{3}{2}]$ are not defined so how do we find $z[1]$ and $z[3]$??

One solution is to <u>INTERPOLATE</u>

A simple interpolation is

$$z[n] = \begin{cases} b[n/2], & n \text{ even} \\ 1/2\,\{b[(n-1)/2] + b[(n+1)/2]\}, & n \text{ odd} \end{cases}$$

Interpolation can be used in a simple compression scheme – just send every other sample and fill in missing values.

Sometimes it works well and sometimes it doesn't.

Back to our example,
Form $z_1[n] = w_1[\frac{n}{2}]$ and $z_2[n] = w_2[\frac{n}{2}]$. Which do you prefer?

Time Shifting

$$y[n] = x[m]|_{m=n-n_0} = x[n - n_0]$$

Here, $y[n]$ is a time-shifted version of the original signal $x[n]$.

Ex.: Given $x[n] = a^n u[n]$, $|a| < 1$, find and plot $y[n] = x[n - 3]$

Combination of Time Shift and Time Scale

Ex. Find $u[3-n]$,

There are two direct ways to find this (in addition to the book method):

1. Reverse then delay ($x[a(n+\frac{b}{a})]$):

$$\begin{aligned} z[n] &= u[-n] \\ y[n] &= z[n-3] = u[-(n-3)] = u[-n+3] \end{aligned}$$

2. Advance then reverse ($x[an+b]$):

$$\begin{aligned} w[n] &= u[n+3] \\ y[n] &= w[-n] = u[-n+3] \end{aligned}$$

When we have a combination of shifting and scaling or reversing, you need to be careful. For example, if we try to form:

$$z[n] = x[3-2n] = x[-2(n-\frac{3}{2})],$$

What does it mean to shift a signal by $\frac{3}{2}$?!! Method 1 does not work here; in other cases method 2 does not work. To be safe, plug in values in a table instead or as a check.

n	$z[n]$	$x[3-2n]$
0	$z[0]$	$x[3]$
1	$z[1]$	$x[1]$
2	$z[2]$	$x[-1]$
-1	$z[-1]$	$x[5]$
-2	$z[-2]$	$x[7]$

<u>Ex.</u> Let $x[n] = 2u[n+2]$. Find $z[n] = x[3-2n]$.

Ex. Let $y[n] = a^n u[n]$, where $a > 1$. Find and plot $z[n] = y[-2n + 2]$.

Ex.
Find $y[n] = x[2-2n]$:

$$x[2-2n] = x[-2(n-1)]$$

$v[n] = x[-2n]$, then delay by 1. Or, just plug in values of n in a table.

They cover amplitude transformations in the book, which you should review but it is similar to continuous time. Basically, given $y[n] = Ax[n] + B$, if $A < 0$, you get amplitude reversal; $|A|$ controls the amplitude scaling; and B controls amplitude shifting. In addition, you should be able to do several other amplitude operations on signals, including: finding magnitude and phase (or real and imaginary parts) on complex signals, and adding or multiplying two signals. Remember: amplitude operations require point-by-point repetition of an operation.

Example: Find $x[n] = (u[n+1] - u[n-5])(nu[2-n])$

9.3 Characteristics of Discrete-Time Signals

Even and Odd Signals

Any discrete-time signal can be expressed as the sum of an even signal and an odd signal:

$$x[n] = x_e[n] + x_o[n]$$

$$\begin{aligned} \text{Even:}\;\; x_e[n] &= x_e[-n] \\ \text{Odd:}\;\; x_o[n] &= -x_o[-n] \end{aligned}$$

$$\begin{aligned} x_e[n] &= \frac{1}{2}(x[n] + x[-n]) \\ x_o[n] &= \frac{1}{2}(x[n] - x[-n]) \\ x[n] &= x_e[n] + x_o[n] \end{aligned}$$

Ex. Given $x[n]$, find $x_e[n]$ and $x_o[n]$.

Signals Periodic in Discrete Time

How do we tell if a discrete-time signal $x[n]$ is periodic? That is, given n and N are integers, is there some period $N > 0$ such that

$$x[n] = x[n+N]?$$

Let's examine a signal that did not necessarily come from sampling a continuous time signal:

$$x[n] = Ca^n$$

Let $a = e^{j\Omega_0} \Rightarrow$ then $x[n] = Ce^{j\Omega_0 n}$ is a complex exponential.

If $x[n]$ is periodic, then $x[n] = x[n+N]$ and

$$Ce^{j\Omega_0 n} = Ce^{j\Omega_0 (n+N)} = Ce^{j\Omega_0 n}e^{j\Omega_0 N}$$

which implies $e^{j\Omega_0 N} = 1$.

When does this happen? Only if $\Omega_0 N$ is an integer multiple of 2π, because $e^{j2\pi} = 1$ and so, $e^{j2\pi k} = 1$ for k an integer. Therefore,

$$\Omega_0 N = 2\pi k \quad \text{or} \quad \frac{\Omega_0}{2\pi} = \frac{k}{N}$$

$\frac{\Omega_0}{2\pi}$ is the normalized frequency – it must be a RATIONAL number for the complex sinusoid to be periodic and there are k cycles of the sinusoid in N samples.

If $\frac{\Omega_0}{2\pi}$ is irrational, then $e^{j\Omega_0 n}$ is not periodic and we never get the samples repeated no matter how many samples we see. The same is true for sinusoids (since they are made of complex exponentials).

[1]

[1] Note that the frequency Ω_0 is not always the same as the fundamental frequency. Since the period N must be an integer for a discrete-time signal, the fundamental frequency is $2\pi/N = \frac{\Omega_0}{k}$, which is the same as Ω_0 only for cases where $k = 1$. The *fundamental period* can be found as

$$N = \frac{2\pi k}{\Omega_0} \quad \text{where } k \text{ is the smallest integer such that } N \text{ is an integer}$$

or by normalizing frequency and reducing to the simplest ratio of integers

$$\frac{\Omega_0}{2\pi} = \frac{k}{N}$$

Ex. Determine which of the signals below are periodic. For the ones that are, find the fundamental period and fundamental frequency.

1. $x_1[n] = e^{j\frac{\pi}{6}n}$

2. $x_2[n] = \sin(\frac{3\pi}{5}n + 1)$

3. $x_3[n] = \cos(2n - \pi)$

4. $x_4[n] = \cos(1.2\pi n)$

5. $x_5[n] = e^{-j\frac{n}{3}}$

There is a major difference between discrete and continuous time. For continuous time, distinct values of frequency produce distinct sinusoids. For discrete-time, complex exponentials and sinusoids with frequency Ω_0 and $\Omega_0 + 2\pi$ are indistinguishable.

Example: for integer n

$$e^{j\frac{\pi}{4}n} = e^{j\frac{9\pi}{4}n} = e^{-j\frac{7\pi}{4}n} = e^{j(200\pi+\frac{\pi}{4})n}$$

but for real t

$$e^{j\frac{\pi}{4}t} \neq e^{j\frac{9\pi}{4}t} \quad \text{etc.}$$

$\Rightarrow$ So only consider frequency interval of length 2π such as $[0, 2\pi)$, $[-\pi, \pi)$. We'll visit this again with the Discrete Time Fourier Transform, but in the mean time we'll highlight the difference by using the notation Ω for frequency (vs. ω for continuous time).

9.4 Common Discrete-Time Signals

$$x[n] = Ca^n \quad C \text{ and } a \text{ can be complex}$$

1. If C and a are real, then $x[n]$ is a real exponential.

   (a) $a > 1$

   This is a growing exponential.
   An example is a GOOD investment.
   C =initial investment, 10% interest rate, $a = 1 + 0.1 = 1.1$.

   (b) $0 < a < 1$

   BAD investment.

(c) $-1 < a < 0$

Alternates and decays (Pendulum).

(d) $a < -1$

Alternate and grows (Teenage mood swings).

2. If both C and a are complex, i.e. $C = Ae^{j\phi}$ and $a = e^{\Sigma_0 + j\Omega_0}$, then we get

$$\begin{aligned} x[n] &= Ca^n = Ae^{j\phi}e^{(\Sigma_0 + j\Omega_0)n} = Ae^{\Sigma_0 n}e^{j(\Omega_0 n + \phi)} \\ &= Ae^{\Sigma_0 n}\cos(\Omega_0 n + \phi) + jAe^{\Sigma_0 n}\sin(\Omega_0 n + \phi) \end{aligned}$$

This will be a damped complex exponential, i.e. it will have sinusoidal real and imag components that either grow or decay depending on whether $\Sigma_0 < 0$ (decays) or $\Sigma_0 > 0$ (grows)

The notation above follows what we used in continuous time, e.g.

$$x(t) = Ae^{j\phi}e^{(\sigma + j\omega)t} = Ae^{\sigma t}e^{j(\omega t + \phi)}$$

but in discrete time it is often more convenient to put a directly in polar form rather than write it as a complex exponential. In this case:

$$\begin{aligned} x[n] &= Ca^n = Ae^{j\phi}(re^{j\Omega_0})^n = Ar^n e^{j(\Omega_0 n + \phi)} \\ &= Ar^n\cos(\Omega_0 n + \phi) + jAr^n\sin(\Omega_0 n + \phi) \end{aligned}$$

where r determines whether it is decaying ($|r| < 1$) or growing ($|r| > 1$) and Ω_0 determines the oscillation.

9.5 Discrete-Time Systems

Recall: A system is an operator on signals, so a discrete-time system is an operator on discrete-time signals.

We will see examples of different discrete-time systems throughout the quarter and on your MATLAB exercises. A particularly important class of LTI systems that we will work with are filtering systems, which include:

- Low-pass filters – systems that remove high frequencies in an input signal
- High-pass filters – systems that remove low frequencies in an input signal
- Band-pass filters – systems that only pass frequencies in a certain frequency band

Another example of a discrete-time system is the Euler integrator from the textbook. It has the equation:

$$y[n] = y[n-1] + Hx[n-1].$$

Here, $x[n]$ is the input to the system and $y[n]$ is the output of the system.

We can also write:

$$y[n] = T(x[n])$$

which represents a transformation. Given the input $x[n]$, we solve equations to obtain the output $y[n]$.

Another simple example of a system, described by a difference equation, is a digital filter:

$$y[n] = (1-\alpha)y[n-1] + \alpha x[n]$$

where $0 < \alpha < 1$. (We will later see that this is a simple low-pass filter.)

9.6 Properties of Discrete-Time Systems

We'll see that these properties are very similar to those in continuous time.

Memory

Condition is same as in continuous time. $y[n_0] = f(x[n_0])$ alone $\rightarrow$ system is memoryless. Otherwise, the system has memory, meaning that its output depends on inputs other than just at the time of the output.

$y[n] = x[n] + 5$ is memoryless
$y[n] = (n+5)x[n]$ is memoryless
$y[n] = x[n+5]$ has memory

Invertibility

Formal definition: A system T has an *inverse* T_i if when cascaded with T gives the identity system (the output of the two systems is the original input):

$$T_i[T(x[n])] = x[n]$$

Unit advance and Unit delay are Inverses.

$$\begin{aligned} T &: \quad y[n] = x[n+1] \\ T_i &: \quad x[n] = y[n-1] \end{aligned}$$

Accumulator and First Difference are Inverses.

$$\begin{aligned} T &: \quad y[n] = \sum_{k=-\infty}^{n} x[k] \\ T_i &: \quad x[n] = y[n] - y[n-1] \end{aligned}$$

A rectifier $y[n] = |x[n]|$ is *not* invertible.

For simple systems, we can easily find the inverse and thereby show invertibility, or we can find two inputs that give the same output and thereby show that the system is not invertible. For more complex systems, the transforms that we will learn later will be useful for determining if a system is invertible.

Causality

Formal definition: A system is causal if output $y[n]$ at $n = n_0$ depends only on $x[n]$ for $n \leq n_0$. The output DOES NOT depend on future inputs but only on past and present inputs. Test this by looking at the time inside the $x[\cdot]$ relative to the time inside $y[\cdot]$.

Intuition: A causal system does not laugh before it is tickled. The output does not start before the input. (Note: having the output be non-zero does not always mean that the output has "started" – consider $y(t) = 1 + x(t)$.)

All real-time physical systems are causal. BUT, you can have a noncausal system – processing images (and other signals) on a computer for later viewing (or playing).

Memoryless implies causal but not vice versa.

Examples:

Bounded-Input Bounded-Output (BIBO) Stability

Formal definition: A system is BIBO stable if an input $|x[n]| \leq B_1, \forall n$ produces an output $|y[n]| \leq B_2, \forall n$.

Intuition: Reasonable (well-behaved) inputs do not cause the system to blow up.

Examples:

Unit delay $y[n] = x[n-1]$ is stable

$y[n] = \cos(x[n])$ is stable

Accumulator is not stable

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

We will see later that this system has an *impulse response* of $h[n] = u[n]$ and we will see this is not BIBO stable.

Time-Invariance

Formal definition: A system is time-invariant if a time shift in the input only results in the same time shift in the output. Mathematically, we can write this as:

$$\begin{aligned} T[x[n]] &= y[n] \\ T[x[n-n_0]] &= y[n-n_0] \end{aligned}$$

Intuition: A system is time-invariant if its behavior doesn't change with time.

Formal test for time-invariance:

Examples:

1. $y[n] = x[2n]$

2. $y[n] = \sum_{k=-\infty}^{n} x[k]$

3. $y[n] = \sum_{k=0}^{n} x[k]$

4. $y[n] = nx[n]$

5. $y[n] = x[n]u[n]$

Linearity

Formal definition: A system is linear if both additivity and scaling hold:

$$T[x_1[n]] = y_1[n] \text{ and } T[x_2[n]] = y_2[n] \Rightarrow$$
$$T[ax_1[n] + bx_2[n]] = ay_1[n] + by_2[n]$$

Special case test for non-linear systems: Zero input must produce zero output due to scaling property:

$$T[ax_1[n]] = ay_1[n]$$

Let $a = 0$, then $T[0] = 0$.

General (formal) test for linearity:

Examples:

Chapter 10 – Discrete-Time Linear Time-Invariant Systems

We will study discrete-time systems that are both linear and time-invariant and see that their input/output relationship is described by a discrete-time convolution.

10.1 Impulse Representation of Discrete-Time Signals

We can write a signal $x[n]$ as:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

or

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

which is writing $x[n]$ as a series of impulse functions shifted in time, all scaled with weights $x[k]$. We will see this again when we show that the I/O relationship of a DT LTI system is a DT convolution.

10.2 Convolution for Discrete-Time Systems

Using the result from 10.1:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

and the fact that the system is linear plus knowledge that the response to $\delta[n-k]$ is $h_k[n]$

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\ y[n] &= \sum_{k=-\infty}^{\infty} x[k]h_k[n] \end{aligned}$$

Due to Time-Invariance, we get $h_k[n] = h[n-k]$

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h_k[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k]
\end{aligned}
$$

CONVOLUTION!!

Note that $h[n]$ is the impulse response.
We get the Convolution Equation:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The output of an LTI system is the input convolved with the impulse response where $h[n]$ is the impulse response.

Convolution is useful in image processing– a KERNEL is passed over each pixel of the image to effect a desired image processing operation such as filtering, edge detection, etc.

As for CT, DT convolution is commutative:
Let $m = n - k$ in above equation,

$$\sum_{n-m=-\infty}^{\infty} x[n-m]h[m] \Rightarrow \sum_{-m=-\infty}^{\infty} h[m]x[n-m] \Rightarrow$$

$$\sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]$$

It doesn't matter which signal gets flipped.

Steps to perform convolution:

1. Time reverse $h[k]$ and shift by n to form $h[n-k]$ (flip and shift)
2. Rewrite $x[n]$ as $x[k]$
3. Multiply $x[k]$ and $h[n-k]$ for all values of k
4. Sum up $x[k]h[n-k]$ over all k to get $y[n]$
5. Do for all values of n

I HAVE TWO IMPORTANT RULES FOR PERFORMING A CONVOLUTION:

1. FLIP THE EASY FUNCTION!
2. DRAW A PICTURE!

Ex.

Find $x[n] * h[n] = y[n]$.
Note: $N_y = N_x + N_h - 1$,
where N_i is the nonzero length of $i[n]$.

Ex. Find $x[n] * \delta[n-n_0]$ $\Rightarrow$ This is CONVOLUTION WITH DISCRETE-TIME IMPULSE $\Rightarrow$ Result is: Convolving a function with an impulse shifts the function to where the impulse is

What is $x[n] * \delta[n]$?

Ex. Find $y[n] = x[n] * h[n]$ where $x[n] = a^n u[n]$ and $h[n] = u[n]$. Try it both ways (first flip $x[n]$ and do the convolution and then flip $h[n]$ and do the convolution). Which method do you prefer?

Ex. Find $y[n] = x[n] * h[n]$ where:

More on DT convolution
<u>Ex.</u> $x[n] = h[n] = u[n]$. Find $y[n] = x[n] * h[n]$.

1. Do it Graphically:

2. Use convolution equation (less preferred by me)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} u[k]u[n-k]$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} u[n-k] \text{ since } u[k] = 0,\ k < 0$$

Now,

$$u[n-k] = 0,\ n-k < 0 \text{ or } k > n \Rightarrow y[n] = \sum_{k=0}^{n} (1) = n+1$$

BUT what values of n is this good for?

$$u[k] = 0,\ k < 0 \text{ and } u[n-k] = 0,\ k > n$$

$\Rightarrow$ only good for $0 < k \leq n \Rightarrow n \geq 0$.

Ex.

$$x[n] = b^n u[n]$$

$$h[n] = a^n u[n+2]$$

where $a \neq b$

Find $y[n] = x[n] * h[n]$.

Ex. Compute output of $x[n] = u[-n]$ to system with impulse response -

$$h[n] = a^n u[n-2],\ \ |a| < 1$$

Answer is:

$$\frac{a^2}{1-a} u[2-n] + \frac{a^n}{1-a} u[n-3]$$

Ex. Given $x[n] = u[n]$ and $h[n] = a^n u[n+2]$, find

$$y[n] = x[n] * h[n]$$

Ex. Given $x[n] = u[-n+2]$ and $h[n] = a^n u[-n]$, find

$$y[n] = x[n] * h[n]$$

The output should be left-sided.

Here are some examples of discrete-time impulse responses:

Unit delay: $h[n] = \delta[n-1]$
Unit advance: $h[n] = \delta[n+1]$
Accumulator: $h[n] = u[n]$
Edge detector: $h[n] = \delta[n] - \delta[n-1]$

Step Response of a Discrete-Time System

The step response of an LTI system is just the response of the system to an input equal to unit step. We can denote this as $s[n]$.

Ex. Compute DT step response of an LTI system with $h[n] = a^n u[n-2]$.

Ex. Given $h[n] = a^n u[n]$ and $x[n] = 3u[n] - 2u[n-3] + u[n-6] - 2u[n-8]$, use SUPERPOSITION to find $y[n] = x[n] * h[n]$.

10.3 Properties of Discrete-Time LTI Systems

The I/O characteristics of an LTI system are completely characterized by its impulse response $h[n]$ (and the output is just the convolution of the input with $h[n]$). We can derive properties of LTI systems based on this by putting constraints on $h[n]$.

Memoryless Systems

The impulse response of a memoryless LTI system can only have the form

$$h[n] = K\delta[n].$$

Anything else would cause inputs other than at the present time to appear in the output.

Invertible Systems

An LTI system with impulse response $h[n]$ is invertible if there exists another function $h_i[n]$ such that

$$h[n] * h_i[n] = \delta[n]$$

Ex. What is the inverse of $h[n] = 3\delta[n + 5]$?

Causality

$$h[n] = 0, \; n < 0$$

so that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

depends only on past and present values of input.

Ex. We can also see the requirements for an LTI system to be causal through convolution. Let $h_1[n] = u[n]$ and form $h_1[n-k]$. What inputs appear in the output?

Now let $h_2[n] = u[n+2]$ and form $h_2[n-k]$. How does this relate to causality?

Stability

Requirement for Bounded-input/Bounded-output.

Stability is

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty,$$

That is, the impulse response must be absolutely summable for a system to be BIBO stable.

This is because:

Given $|x[n]| \leq M$ for all n,

examine the magnitude of the output. It must be finite for BIBO stability.

$$|y[n]| = |\sum_{k=-\infty}^{\infty} x[n-k]h[k]| \leq \sum_{k=-\infty}^{\infty} |x[n-k]h[k]| = \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq$$

$$\sum_{k=-\infty}^{\infty} M|h[k]| = M \sum_{k=-\infty}^{\infty} |h[k]|$$

Since $M < \infty$, we only need $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ for $y[n]$ to be finite.

Ex. Is

$$h[n] = (\frac{1}{3})^n u[n]$$

BIBO stable?

More examples:

1. $h_1[n] = u[n]$ (the accumulator)

2. $h_2[n] = 3^n u[n]$

3. $h_3[n] = (3)^n u[-n]$

4. $h_4[n] = \cos(\frac{\pi}{3}n)u[n]$

5. $h_5[n] = u[n+2] - u[n]$

Unit-Step Response

$$\begin{aligned} x[n] &= u[n] \\ s[n] &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k] \end{aligned}$$

Can get $h[n]$ from $s[n]$ as:

$$h[n] = s[n] - s[n-1]$$

Ex. Given we already determined that for an impulse response $h[n] = a^n u[n]$, $s[n] = u[n] * a^n u[n] = \frac{1-a^{n+1}}{1-a} u[n]$, show that you can obtain the impulse response back from the step response. You might find it helpful to remember that $u[n-1] = u[n] - \delta[n]$.

Summary of DT LTI Systems.

1. System Attributes,

   Memory, linearity, TI, Causality, Stability, Invertibility.

   Saw how to determine attribute from impulse response (except inverse).

2. Saw how an LTI system has its I/O relationship described by convolution.

3. Superposition–Break an input down into basis functions for which it is easy to calculate system response.

   Ex. Impulses, step functions, exponentials.

4. Can get step response from impulse response and vice versa.

Continuous Time	Discrete Time
$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$	$u[n] = \sum_{k=-\infty}^{n} \delta[k]$
$s(t) = \int_{-\infty}^{t} h(\tau) d\tau = h(t) * u(t)$	$s[n] = \sum_{k=-\infty}^{n} h[k] = h[n] * u[n]$
$\delta(t) = \frac{d}{dt} u(t)$	$\delta[n] = u[n] - u[n-1]$
$h(t) = \frac{d}{dt} s(t)$	$h[n] = s[n] - s[n-1]$

10.4 Difference-Equation Models

LTI discrete-time systems are usually modeled by linear difference equations with constant coefficients. For example, a digital filter is modeled by a difference equation.

An example of a difference equation is:

$$y[n] = x[n] + x[n-1] + x[n-2].$$

A general Nth order $(N \geq M)$ linear difference equation with constant coefficients (LCCDE) is:

$$a_0 y[n] + a_1 y[n-1] + \ldots + a_{N-1} y[n-N+1] + a_N y[n-N] =$$

$$b_0 x[n] + b_1 x[n-1] + \ldots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

which we can write as:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where a_k and b_k are real constants.

An important case to be familiar with is the first-order system

$$y[n] = ay[n-1] + bx[n]$$

in which the output is a function of a delay of only one time unit.

The Classical method for the solution is to express the output $y[n]$ as the sum of *complementary* or *natural* $(y_c[n])$ and *particular* or *forced* $(y_p[n])$ solutions:

$$y[n] = y_c[n] + y_p[n]$$

Natural response The natural response is the solution to the homogeneous equation:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

where $a_0 \neq 0$.

We assume solutions of the form $y_c[n] = Cz^n$.

We can see that:

$$y_c[n] = Cz^n, y_c[n-1] = Cz^{n-1} = Cz^{-1}z^n, \ldots$$

$$y_c[n-N] = Cz^{n-N} = Cz^{-N}z^n$$

and substituting in the homogeneous equation yields:

$$(a_0 z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N) C z^{-N} z^n = 0$$

and we get the characteristic equation:

$$a_0 z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N = a_0 (z - z_1)(z - z_2) \ldots (z - z_N) = 0.$$

Clearly, N values of z satisfy this equation.

The solution is of the form:

$$y_c[n] = C_1 z_1^n + C_2 z_2^n + \ldots + C_N z_N^n$$

assuming there are no repeated roots (which is all we will cover).

Ex. Given a first-order difference equation

$$y[n] + .2y[n-1] = x[n]$$

find its homogeneous solution. Your answer should be in terms of a constant C.

Forced response The forced response $y_p[n]$ solves the equation

$$\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

The form of the solution is determined by the input $x[n]$. For an exponential input $x[n] = Aa^n$, the solution would be $y_p[n] = Pa^n$ where A, a, and P are constants.

Ex. For the previous example, given an input $x[n] = 9(.7)^n$, find the particular solution $y_p[n]$.

Ex. Now, assuming that the system is initially at rest, i.e., initial conditions of 0 ($y[0] = 0$), solve for the constant C in your overall solution $y[n] = y_c[n] + y_p[n]$.

Ex. Given

$$y[n] - .3y[n-1] = x[n]$$

with $y[-1] = 0$ and $x[n] = (.6)^n$, find $y[n]$.

10.5 Terms in the Natural Response

Recall that the natural solution was

$$y_c[n] = C_1 z_1^n + C_2 z_2^n + \ldots + C_N z_N^n$$

where z_i is the root of the characteristic equation.

$$a_0 z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N = a_0(z - z_1)(z - z_2)\ldots(z - z_N) = 0.$$

Each general term is $C_i z_i$ where z_i^n is a system mode. The root z_i can be either real or complex. Its value will determine if the overall system is BIBO stable or not.

Assume we have a *causal* LTI system. The solution is of the form

$$y[n] = y_c[n] + y_p[n].$$

Since $y_p[n]$ is of the form $x[n]$, if the input $x[n]$ is bounded, then $y_p[n]$ will also be bounded.

Let's examine

$$y_c[n] = C_1 z_1^n + C_2 z_2^n + \ldots + C_N z_N^n.$$

Clearly, as long as all roots (also called *poles*) of this equation, $|z_i[n]| < 1$, then each term in $y_c[n]$ will be bounded.

Our condition for stability of a causal LTI system is that all roots of the system characteristic equation lie within the unit circle in the z-plane, that is, $|z_i| < 1, \forall i$.

You will have the opportunity to examine this further when you do Laboratory 3 – be sure to see what happens to the impulse response of the system when roots lie outside the unit circle.

Ex. Given a causal system described by the difference equation

$$y[n] - 2.5y[n-1] + y[n-2] = x[n]$$

determine if the system is BIBO stable.

Ex. Given a causal system described by the difference equation

$$y[n] - 1.25y[n-1] + .375y[n-2] = x[n]$$

determine if the system is BIBO stable.

10.6 Block Diagrams

We will not cover this section.

10.7 System Response for Complex-Exponential Inputs

Given an input $x[n] = Xz^n$ to a BIBO stable LTI system modeled by an Nth order linear difference equation with constant coefficients, we will examine the steady-state system response. Here, X and z are complex.

If the system is stable, then the natural system response will die out. The forced (steady-state) response of the system to this input is of the same form as the input, i.e.

$$y_p[n] = y_{ss}[n] = Yz^n.$$

From the difference equation describing the system,

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k],$$

plugging in $x[n]$ and $y_{ss}[n]$, we get:

$$\sum_{k=0}^{N} a_k Y z^{n-k} = \sum_{k=0}^{M} b_k X z^{n-k}$$

or

$$Y \sum_{k=0}^{N} a_k z^{n-k} = X \sum_{k=0}^{M} b_k z^{n-k}$$

or

$$Y = X \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

which we can write as
$Y = XH(z)$
where

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

is a *transfer function.*

So given an input $x[n] = Xz_1^n$ to such an LTI system, the steady-state response is $y_{ss}[n] = XH(z_1)z_1^n$.

Similar to the difference equation, the transfer function $H(z)$ completely characterizes the LTI system (we can derive the difference equation from $H(z)$ and vice versa).

In general, by superposition, given an input $x[n] = \sum_{k=1}^{M} X_k z_k^n$, the output of the system is $y_{ss}[n] = \sum_{k=1}^{M} X_k H(z_k) z_k^n$.

Finally, if

$$x[n] = \sum_k a_k \phi_k[n]$$

and

$$y[n] = \sum_k a_k \psi_k[n]$$

where $\psi_k[n] = \phi_k[n] * h[n]$ AND $\psi_k[n] = b_k \phi_k[n]$ (input and output basis functions have the same form), then $\phi_k[n]$ is an eigenfunction of the LTI system with eigenvalue b_k.

Analogous to CT, eigenfunctions of DT LTI systems are complex exponentials:

$$\phi[n] = z^n$$

Check: What is $z^n * h[n]$?

$$\psi[n] = \phi[n] * h[n] = z^n * h[n] = \sum_{k=-\infty}^{\infty} z^{n-k} h[k] = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = z^n H(z)$$

where $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ is the eigenvalue. This motivates the Z-transform and the Discrete Time Fourier Transform (DTFT):

- $H(z)$ is known as the z-transform of $h[n]$.
- If $z = e^{j\Omega}$, then we get $H(\Omega)$, the DTFT of $h[n]$.

Ex.

Given an input $x[n] = (\frac{3}{4})^n$, and

$$h[n] = (.5)^n u[n]$$

find its steady-state output

$$y_{ss}[n] = H(\frac{3}{4})(\frac{3}{4})^n$$

is the forced response

YOU FINISH:

Chapter 11 – The Z-Transform

The Z-transform is the Discrete-Time counterpart of the Laplace Transform.

$$\begin{aligned} Laplace &: \; F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \\ Z &: \; F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n} \end{aligned}$$

It is

- Used in Digital Signal Processing
- Used to Define Frequency Response of Discrete-Time System.
- Used to Solve Difference Equations – use algebraic methods as we did for differential equations with Laplace Transforms; it is easier to solve the transformed equations since they are algebraic.

We will see that

1. Lines on the s-plane map to circles on the z-plane.
2. Role of $j\omega$-axis is replaced by unit circle, so
   (a) The DT Fourier Transform exists for a signal if the ROC includes the unit circle.
   (b) A stable system must have an ROC that contains the unit circle.
   (c) A causal and stable system must have poles inside the unit circle.

Aside: You can relate the Z transform and Laplace transform directly when you are dealing with sampled signals:

Take a CT signal $f(t)$ and sample it:

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)$$

The Laplace transform of the sampled signal is

$$\begin{aligned}
\mathcal{L}[f_s(t)] &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT) \right] e^{-st} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(nT)\delta(t - nT) e^{-st} dt \\
&= \sum_{n=-\infty}^{\infty} f(nT) e^{-snT}
\end{aligned}$$

by the sifting property.

Let $f[n] = f(nT)$ and $z = e^{sT}$, then

$$\begin{aligned}
F(z) &= \sum_{n=-\infty}^{\infty} f[n] z^{-n} \\
F(z)|_{z=e^{sT}} &= \sum_{n=-\infty}^{\infty} f[n] e^{-sTn} \\
&= \sum_{n=-\infty}^{\infty} f(nT) e^{-snT} \\
&= \mathcal{L}[f_s(t)]
\end{aligned}$$

Thus, the Z tranform with $z = e^{sT}$ is the same as the Laplace transform of a sampled signal! Of course, if the signal is already discrete, the notion of sampling is unnecessary for understanding and using the Z transform.

11.1 Definitions of Z-Transforms

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

is the bilateral (2-sided) Z-transform. Its inverse Z-transform is defined as:

$$Z^{-1}[H(z)] = h[n] = \frac{1}{2\pi j} \oint H(z)z^{n-1}dz$$

which is a counterclockwise contour integral along a closed path in the z-plane. We will see how to take inverse Z-transforms using tables and partial fraction expansion.

We can also define a *unilateral* Z-transform as

$$H_u(z) = \sum_{n=0}^{\infty} h[n]z^{-n}.$$

IMPORTANT: The textbook uses the notation $Z_b[f[n]]$ and $F_b(z)$ to denote the two-sided or bilateral Z Transform of a function $f(n)$, but since we will only look at the bilateral transform (and NOT the unilateral), we will drop the b subscript here.

11.7 Bilateral Z-Transform

Since our focus is on the Bilateral Z-Transform, let's jump ahead to that section, and then we'll come back to the earlier chapters.

The bilateral (2-sided) Z-transform.

$$H_b(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

is for 2-sided, anticausal, and causal signals (i.e. all signals).

Note that, whereas for Laplace Transform we considered where the integral converges, here we consider where the sum converges.

We must consider the Region of Convergence (ROC) of the Z-transform for the bilateral Z-transform because left-sided and right-sided time functions will have the same Z-transform and only the ROC will distinguish between the two possible time functions.

Remember:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}, \quad |a| < 1$$

You'll use this a lot!

Ex. Find the Z tranforms of

$$x_1[n] = a^n u[n] \quad \text{and} \quad x_2[n] = -(a^n)u[-n-1]$$

and plot the ROCs and pole/zero diagrams.

We see that we must specify the ROC for the bilateral Z- transform to be unique.

Definitions and Regions of Convergence

- $x[n]$ is right-sided if $x[n] = 0, n < n_0$
- $x[n]$ is left-sided if $x[n] = 0, n > n_0$

We can write

$$X(z) = \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

1. We've seen that **right-sided** signals have an ROC of the form $|z| > r_{max}$, i.e., it converges outside the largest magnitude pole.

   (Infinite Egg White)

   Examine for right-sided $x[n]$

$$X(z) = \sum_{n=n_0}^{\infty} x[n]z^{-n}$$

$$X(z) = \sum_{n=n_0}^{\infty} x[n]\left(\frac{1}{z}\right)^n$$

   As $n \to \infty$, need $(1/z)^n \to 0$ for sum to converge.

   This will happen for values of z outside rather than inside the pole, i.e. $|z| > r_{max}$.

   What about $z = \infty$?

   If $x[n]$ is not causal but is still right-sided, e.g. $x[n] = u[n+1]$, then

$$X(z) = \sum_{n=-1}^{\infty} z^{-n} = z + \sum_{n=0}^{\infty} z^{-n}$$

   Will not converge at $z = \infty$, and we won't include it in the ROC.

   Thus we can tell if a system is causal from the ROC of the Z-transform of its impulse response.

$$\begin{aligned} |z| > r_{max} &\Rightarrow \text{CAUSAL} \\ \infty > |z| > r_{max} &\Rightarrow \text{right-sided but not causal} \end{aligned}$$

2. **Left-sided** signals have ROC of form $|z| < r_{min}$, i.e., it converges INSIDE circle $|z| < r_{min}$ (EGG YOLK).

   Examine for left-sided $x[n]$

$$X(z) = \sum_{n=-\infty}^{n_0} x[n]z^{-n}$$

   As $n \to -\infty$, need $(1/z)^n \to 0$ or $z^{\infty} \to 0$

   This happens for values of z inside rather than outside the poles.

   <u>What about $z = 0$</u> ?

   If $x[n]$ is left-sided but not strictly anticausal

   ($x[n] = 0$ for $n > n_0 > 0$ but $x[n_0] \neq 0$)

   e.g. $x[n] = u[-n+1]$, then

$$X(z) = \sum_{n=-\infty}^{1} z^{-n} = z^{-1} + \sum_{n=0}^{\infty} z^n$$

   does not converge at $z = 0$ so don't include $z = 0$ in the ROC.

3. **2-sided** signals have ROC of the form

$$r_1 < |z| < r_2 \text{ (BAGEL OR DONUT)}$$

4. **Finite Duration** $x[n]$ has ROC of entire z-plane except possibly $z = 0$ or $z = \infty$

$$\delta[n-1] \leftrightarrow z^{-1}, |z| > 0$$

$$\delta[n+1] \leftrightarrow z, |z| < \infty$$

FACT: An ROC must contain the unit circle for stability – this holds for causal, anticausal, and two-sided signals.

Ex. Find the Z-Transform of $x[n] = a^{|n|}$ for $|a| < 1$.

Ex. Find the Z-Transform of

$$x[n] = 3^n u[-n-1] + 4^n u[-n-1].$$

Ex. Find the Z-transform of $\frac{1}{2}\delta[n-1] + 3\delta[n+1]$.
What is its ROC?

Ex. Find the Z-transform of

$$x[n] = (.5)^n u[n-1] + 3^n u[-n-1].$$

Would this system be BIBO stable?

Ex. Find the Z-transform of $x[n] = r^n \sin(bn)u[n]$ using Euler's rule.

Note that the examples in Section 11.3 and in the table on page 489 are for a unilateral Z-transform but if you add a $u[n]$ to all the time functions, you will get the same answer as for the bilateral transform. You can derive all these transform pairs for practice taking Z-transforms.

Insights from the Pole-Zero Plot and ROC

Things that you can tell about a signal from its pole-zero plot (and ROC):

- When the ROC includes the unit circle, then the signal is absolutely summable. (If the signal is an impulse response $\Rightarrow$ the system is stable.)

- A pole on the positive real axis corresponds to a simple decaying or growing function (of form a^n for a pole at $z = a$).

- Poles off the positive real axis correspond to an oscillating signal where the frequency of oscillation is the the angle from the positive real axis. (Poles on the negative real axis have an angle of π, so the frequency of oscillation is π, as in $(-1)^n$.) When the poles are ...
  - on the unit circle $\Rightarrow$ sinusoidal functions with constant amplitude
  - *not* on the unit circle $\Rightarrow$ sinusoidal functions with a decaying (or growing) envelope (rate of decay/growth depends on the distance from the pole to the origin).

- Poles and zeroes must come in complex conjugate pairs for the signal to be real (consequence of the Z-transform property: $x[n]^* \longleftrightarrow X^*(z^*)$).

Z-Transform Properties

Sections 11.4 and 11.5 focus on the unilateral Z tranform in discussing properties, but as it turns out most of the properties are similar for the two cases. Since they are easier and more general for the bilateral case, we'll focus on those. We discuss the most important ones here:

- Linearity:

  $$ax[n] + by[n] \longleftrightarrow aX(z) + bY(z)$$

  where the new ROC $R' \supset R_x \cap R_y$.

- Time shift:

  $$x[n - n_0] \longleftrightarrow z^{-n_0}X(z)$$

  where the new ROC is the same as R_x with the possible addition or deletion of the origin or infinity.

- Convolution:

  $$y[n] = x[n] * h[n] \longleftrightarrow X(z)H(z)$$

  where the new ROC $R_y \supset R_x \cap R_h$.

Linearity and the time shift property will be useful for LCCDE systems, and the convolution property lets us avoid discrete-time convolutions. We'll use these properties a lot.

Convolution in Time

$$y[n] = x[n] * h[n] \leftrightarrow \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x[k]h[n-k]]z^{-n}$$

because we have a Z-transform. Switching the order of the summations (OK except for pathological cases), we get:

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]z^{-n}$$

Now, let $m = (n-k)$ and we get:

$$= \sum_{k=-\infty}^{\infty} x[k][\sum_{m=-\infty}^{\infty} h[m]z^{-(m+k)}]$$

$$= \sum_{k=-\infty}^{\infty} x[k]z^{-k} \sum_{m=-\infty}^{\infty} h[m]z^{-m} = X(z)H(z)$$

The new ROC will depend on both the poles in $X(z)$ and $H(z)$, giving $R_x \cap R_h$ since the ROC cannot include poles. However, if one transform has a zero that cancels a pole of the other then the ROC can be bigger, hence $R_y' \supset R_x \cap R_h$.

11.6 LTI System Applications

Transfer Functions The Z-transform properties are particularly useful when you have an LTI system described by an LCCDE.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

We can use this to determine outputs of LTI systems by multiplying the Z-transform with the input with $H(z)$ to get the Z-transform of the output. Then we can recover the time domain output using the *Inverse Z-transform.*

Ex. Given a difference equation,

$$y[n] - .3y[n-1] = x[n]$$

find the Z-transform of the equation and then find the response $Y(z)$ of the system to an input $x[n] = (.6)^n u[n]$.

What if you wanted to find the response in the time domain?

$\Rightarrow$ We can use **Partial Fraction Expansion** to invert the Z-transform.

As we saw for Laplace Transforms,

$$Y(z) = \frac{N(z)}{D(z)} = \sum_{k=1}^{N} \frac{r_k z}{z - p_k}$$

$$p_k = \text{pole} \quad r_k = \text{residue}$$

where

$$r_k = [\frac{Y(z)}{z}(z - p_k)]|_{z=p_k}$$

Then use tables to invert the Z-transform, e.g.

$$a^n u[n] \leftrightarrow \frac{z}{z-a}$$

Back to our previous example ...

Ex. Find Inverse Z-Transform of

$$X(z) = \frac{2z^2 - 5z}{(z-2)(z-3)}, |z| > 3$$

Expand:

$$\frac{X(z)}{z} = \frac{2z-5}{(z-2)(z-3)}$$

Ex. Given $h[n] = a^n u[n]$ ($|a| < 1$) and $x[n] = u[n]$, find $y[n] = x[n] * h[n]$.

What if $x[n] = u[n-2]$?

Ex. Find the output $y[n]$ to an input $x[n] = u[n]$ and an LTI system with impulse response

$$h[n] = -3^n u[-n-1].$$

Another method to invert Z-transforms is the **Power Series Expansion**. Using

$$\delta[n-k] \longleftrightarrow z^{-k}$$

$$\begin{aligned} X(z) &= \textstyle\sum_{k=0}^{\infty} x[k]z^{-k} &&= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \\ x[n] &= \textstyle\sum_{k=0}^{\infty} x[k]\delta[n-k] &&= x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \end{aligned}$$

So if you can expand $X(z)$ like this as a series in z^{-1}, you can pick off $x[n]$ as the coefficients of the series.

Ex. Find the Inverse Z-Transform of

$$X(z) = 1 + 2z^{-1} + 3z^{-2}$$

Ex. Find the Inverse Z-Transform of

$$X(z) = \frac{z}{z - 1/2}, \ |z| > 1/2$$

Divide $z - \frac{1}{2}$ into z:

Ex. Find the inverse Z-transform of

$$X(z) = \frac{8z - 19}{z^2 - 5z + 6}$$

$$|z| > 3$$

Ex. Find the inverse Z-transform of

$$H(z) = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1},$$

$$\frac{1}{2} < |z| < 2.$$

Would a system having this Z-transform be BIBO stable?

Ex. Find the inverse Z-transform of

$$W(z) = \frac{z^{-4}}{z^2 - 2z - 3}, |z| > 3$$

Stability

As we saw earlier, for BIBO stability of a causal LTI system, all roots of the system characteristic equation lie within the unit circle in the z-plane.

This is equivalent to stating that all *poles* of the transfer function $H(z)$ must lie within the unit circle on the z-plane. We point out that $H(z)$ does not converge at its poles.

Because causal systems have Regions of Convergence that lie *outside* the largest magnitude pole, an equivalent condition for BIBO stability is that the ROC must contain the unit circle.

Ex. Find the Z-Transform of the unit step $u[n]$. Would an LTI system with $u[n]$ as its system function be BIBO stable?

Ex. Find the Z-transform of $x[n] = (.9)^n u[n]$. Would an LTI system with $x[n]$ as its system function be BIBO stable?

Invertibility

$$h[n] * h_i[n] = \delta[n] \Rightarrow H(z)H_i(z) = 1$$

Ex. Find the inverse system $h_i[n]$ of $h[n] = a^n u[n]$.
Check your results by taking the convolution of $h[n]$ with $h_i[n]$.

Ex. Find the inverse system of $h[n]$ where

$$H(z) = \frac{z - a}{z - b}.$$

For BIBO stability of both systems (assuming they are both causal), where must all poles and zeros of $H(z)$ lie?

Initial and Final Values (11.4)

The initial and final value properties are associated with the unilateral Z-transform, but they hold for the bilateral transform if the time function is known to be 0 for $n < 0$.

Ex. Writing

$$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n} = f[0] + f[1]z^{-1} + f[2]z^{-2} + \ldots,$$

derive the Initial Value theorem by taking $\lim_{z\to\infty} F(z)$.

The Final Value (steady-state value) theorem is

$$\lim_{n\to\infty} f[n] = f[\infty] = \lim_{z\to 1}(z-1)F(z)$$

as long as $f[n]$ has a final value (all poles must be inside unit circle except for possibly a single pole at $z = 1$).

Ex. Find the initial and final values of $f[n]$ where $F(z) = \frac{z}{z-.6}$.

11.5 Additional Properties

Time Scaling Given

$$f[n] \leftrightarrow F(z),$$

show that

$$f[\frac{n}{k}] \leftrightarrow F(z^k)$$

for k a positive integer by doing a change of variables (let $m = \frac{n}{k}$).

This definition of $\frac{n}{k}$ creates additional sample values but we set them to 0. They don't address speedup (i.e. $f[nk]$ in the book). This case loses sample values.

Frequency Response

If we evaluate $H(z)$ at $z = e^{j\Omega}$, i.e., on the unit circle (as long as unit circle is in the ROC), then we get the DTFT $H(\Omega)$, which we call the frequency response. $H(\Omega)$ is periodic with period 2π and you see examples of $H(\Omega)$ in Labs 3 and 5 with the function frevalz01. In frevalz01, though, it's normalized with period 1 $(-.5, .5)$. You will discuss the DTFT in Chapter 12.

Ex. Find the magnitude of $H(\Omega)$ for $h[n] = a^n u[n]$, $0 < a < 1$.

$$|H(\Omega)| = \sqrt{H(\Omega)H^*(\Omega)}$$

Chapter 12 – Fourier Transforms of Discrete-Time Signals

We will study Fourier transforms of discrete-time signals.

12.1 Review of Sampling Continuous Time Signals

Sampling is used, for example, in A/D Conversion, although here we ignore the effects of *quantization*.

Discretize a continuous time signal for CDs, computers, etc.

Define the continuous time impulse train as:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$x(t)$ is the continuous time signal we wish to sample

Let $y(t) = x_s(t) = x(t)p(t)$ be the sampled signal. Then,

$$x_s(t) = y(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$$

$$X_s(\omega) = Y(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

by the multiplication property of the continuous time Fourier Transform. $X(\omega)$ is the Fourier Transform of $x(t)$.

Now find the Fourier Transform of $p(t)$, the infinite impulse train:

$$P(\omega) = \mathcal{F}[\sum_{k=-\infty}^{\infty} \delta(t - kT)]$$

Use the Fourier Transform of periodic signals since the impulse train is a periodic signal (with $\omega_0 = \omega_s$)

$$\mathcal{F}[\sum_k a_k e^{jk\omega_s t}] = \sum_k 2\pi a_k \delta(\omega - k\omega_s)$$

Find a_k for the periodic impulse train:

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T}\int_T \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T} \text{ for all } k \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}[\sum_{k=-\infty}^{\infty} \delta(t - kT)] &= \sum_k \frac{2\pi}{T} \delta(\omega - k\omega_s) \\ &= \sum_k \omega_s \delta(\omega - k\omega_s) \end{aligned}$$

Thus, an impulse train in time has a Fourier Transform that is an impulse train in frequency.

The spacing between pulses in time is T, and the spacing between pulses in frequency is $\frac{2\pi}{T}$.

So increasing the spacing in time decreases the spacing in frequency and vice versa. This is an important result!

Back to $X_s(\omega)$
Let $\omega_s =$ be the sampling frequency

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi} X(\omega) * [\omega_s \sum_k \delta(\omega - k\omega_s)] \\ &= \frac{\omega_s}{2\pi} \sum_k X(\omega - k\omega_s) = \frac{1}{T} \sum_k X(\omega - k\omega_s) \end{aligned}$$

or we get replicated, scaled version of $X(\omega)$.

ω_b is for "bandwidth"

Now, what if $\omega_s - \omega_b < \omega_b$?

We would get overlap of the islands or "aliasing." Therefore, we need $\omega_s - \omega_b > \omega_b$ or $\omega_s > 2\omega_b$ to avoid aliasing.

Sampling Theorem says need $\omega_s > 2\omega_b$ to recover $x(t)$ from its samples–in other words, we need to sample at least twice the highest frequency to avoid aliasing.

We hear music up to $20kHz$ and CD sampling rate is $44.1kHz$.

A dog would need a higher quality CD since they hear higher frequencies.

You can recover $x(t)$ from $y(t)$ by using a low pass filter to recover the center island. They also talk about a zero-order hold system in the textbook.

12.2 Discrete-Time Fourier Transform

For infinite length sequences – in practice, we don't have an infinite amount of data so we'll also study the Discrete Fourier Transform for finite data sequences.

Recall that we wrote the sampled signal $x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$. We calculated its Fourier Transform using the Fourier Transform for a periodic function.

We do the following:

Ex. Find the Continuous Time Fourier Transform of $\delta(t - kT)$.

Ex. Using superposition, find the CT Fourier Transform of $x_s(t)$.

Now, you just calculated that

$$x_s(t) \leftrightarrow \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$$

Let $x(nT) = x[n]$ and make a change of variables $\Omega = \omega T$ (we'll talk more about this later — it relates the discrete-frequency variable Ω to the continuous frequency variable ω via the sampling period T) and we get:

$$\text{DTFT} : X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- Discrete in time but continuous in frequency and periodic
- Spectrum of discrete signal $x[n]$
- We evaluated this in MATLAB 3
- Will use the DTFT to motivate the DFT by taking the DTFT of a windowed data segment
- Will compare the DTFT of a discrete signal $x[n]$ with the Continuous Time Fourier Transform of a sampled continuous time signal $x_s(t) = x(t)p(t)$

Formula to calculate inverse DTFT (this is similar to the Fourier Series):

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$$

where DTFT is periodic in frequency with period 2π. Why? Because $e^{j\Omega}$ is periodic with period 2π.

$$e^{j\Omega} = e^{j(\Omega+2\pi)} = e^{j\Omega}e^{j2\pi} = e^{j\Omega}.$$

Not all DTFTs converge due to the infinite sum.

Ex. 1 Find $X(\Omega)$ where $x[n] = a^n u[n]$, $|a| < 1$. What if $|a| > 1$?

Ex. 2 $y[n] = a^n u[-n]$, $|a| > 1$. Find $Y(\Omega)$.
What if $|a| < 1$?

Ex. 3 Rectangular pulse, $p[n] = u[n] - u[n - N]$. Find $P(\Omega)$.
Show that this filter has a *linear phase* term.

Ex. Find $H(\Omega)$ for

$$h[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3]$$

and show that the filter has a linear phase term.

Z-Transform

We already saw the DTFT as the Z-transform of $x[n]$ evaluated on the unit circle when we discussed the frequency response:

$$X(\Omega) = X(z)|_{z=e^{j\Omega}}$$

If the ROC for the Z-transform contains the unit circle, we can get DTFT from the Z-transform by substitution (compare the DTFT of $a^n u[n]$ with its Z-transform).

We'll see that the DTFT exists in cases where the ROC of the Z-transform does not include the unit circle (e.g. for periodic discrete-time signals) – analogous to the CT Fourier Transform and Laplace Transform.

12.3 Properties of the DTFT

We cover a few here and you can read about the rest in the textbook.

LINEARITY

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(\Omega) + bX_2(\Omega)$$

TIME SHIFT

$$x[n] \longleftrightarrow X(\Omega)$$

$$x[n - n_0] \longleftrightarrow ?$$

$$x[n - n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

So a shift in time causes a linear phase shift in frequency – adds a linear term to the phase of the DTFT.

MODULATION – Frequency Shift

$$x[n] \longleftrightarrow X(\Omega)$$

$$e^{j\Omega_0 n} x[n] \longleftrightarrow ?$$

Modulation causes a shift in frequency.

CONVOLUTION IN TIME

As usual,

$$x_1[n] * x_2[n] \longleftrightarrow X_1(\Omega) X_2(\Omega)$$

Ex. Given $h[n] = a^n u[n], |a| < 1$. Find its inverse system $h_i[n]$.

Ex. $x[n] = (.9)^{|n|}$. Find its DTFT.

MULTIPLICATION OF SIGNALS

This is not in your textbook so you won't be responsible for it

$$x_1[n]x_2[n] \longleftrightarrow \frac{1}{2\pi} X_1(\Omega) \bigotimes X_2(\Omega)$$

where $\otimes$ denotes CIRCULAR CONVOLUTION:

$$X_1(\Omega) \bigotimes X_2(\Omega) = \int_{2\pi} X_1(\theta) X_2(\Omega - \theta)) d\theta$$

$$y[n] = x_1[n]x_2[n] \longleftrightarrow \text{Take its DTFT :}$$

$$Y(\Omega) = \sum_n y[n] e^{-j\Omega n} = \sum_n x_1[n]x_2[n] e^{-j\Omega n}$$

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(a) e^{jan} da$$

$$x_2[n] = \frac{1}{2\pi} \int_{2\pi} X_2(b) e^{jbn} db$$

$$Y(\Omega) = \sum_n \left[\frac{1}{2\pi} \int_{da} X_1(a) e^{jan} da \right] \left[\frac{1}{2\pi} \int_{db} X_2(b) e^{jbn} db \right] e^{-j\Omega n}$$

$$= \left(\frac{1}{2\pi} \right)^2 \int_{da} \int_{db} X_1(a) X_2(b) \sum_n e^{j(a+b)n} e^{-j\Omega n} da db$$

Now,

$$\sum_n e^{j(a+b)n} e^{-j\Omega n}$$

is just the DTFT of

$$e^{j(a+b)n}$$

that is,

$$e^{j(a+b)n} \leftrightarrow 2\pi \delta(\Omega - a - b)$$

So,

$$\begin{aligned} Y(\Omega) &= \frac{1}{2\pi} \int_{\frac{da}{2\pi}} \int_{\frac{db}{2\pi}} X_1(a) X_2(b) \delta(\Omega - a - b) da db \\ &= \frac{1}{2\pi} \int_{2\pi} X_1(a) X_2(\Omega - a) da = \frac{1}{2\pi} X_1(\Omega) \bigotimes X_2(\Omega) \end{aligned}$$

12.4 Transform of Periodic Sequences

Here we study the DTFT of periodic sequences. We'll start by looking at the Fourier Series expansion, analogous to what we did in continuous time. Then we will derive the same result using a different approach that will lead us into the Discrete Fourier Transform for finite length sequences.

Recall that for **continuous time periodic signals**, we found the Fourier transform by first doing a Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{synthesis equation} \qquad (1)$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \qquad \text{analysis equation} \qquad (2)$$

then using the fact that a complex exponential in time transforms to an impulse in the frequency domain

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

and linearity of the Fourier transform, we get that the CTFT of a periodic signal is made up of harmonically-related impulses with area $2\pi a_k$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Discrete-time periodic signals can also be described by a Fourier Series expansion:

$$x[n] = \sum_{k \in <N>} a_k e^{jk\Omega_0 n} \qquad \text{synthesis equation} \tag{3}$$

$$a_k = \frac{1}{N} \sum_{n \in <N>} x[n] e^{-jk\Omega_0 n} \qquad \text{analysis equation} \tag{4}$$

As one would expect, the integral in time goes to a sum. However, there is one more key difference: *the sum in the synthesis equation is finite!* (over an interval the length of a one period).

First, recall that $\Omega_0 = \frac{2\pi}{N}$. Then since $e^{jk\Omega_0 n} = e^{\frac{jk2\pi n}{N}} = e^{\frac{j(k+N)2\pi n}{N}}$ (since $e^{j2\pi n} = 1, \forall n$), the a_k's are periodic with period N and only N terms are needed in the sum.

So, we have expressed periodic $x[n]$ as a finite sum of complex exponentials with discrete frequencies $\frac{2\pi k}{N}$.

The next step is to find the DTFT of $e^{j\Omega_0 n}$. Since this function is not absolutely summable, we need to allow impulses in order for the DTFT to exist. Hoping that the discrete-time case behaves like continuous time, we might guess ...

$$\begin{aligned} CT: e^{j\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega-\omega_0) && (5)\\ DT: e^{j\Omega_0 n} &\longleftrightarrow 2\pi\delta(\Omega-\Omega_0) \quad ???? && (6)\end{aligned}$$

But this can't be right because the DTFT must be periodic! So, instead let's guess:

$$e^{j\Omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega-\Omega_0+2\pi l)$$

Then we can verify that this works with our inverse DTFT equation

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{<2\pi>} X(\Omega)e^{j\Omega n}d\Omega && (7)\\ &= \frac{1}{2\pi}\int_{\Omega_0-\pi}^{\Omega_0+\pi} 2\pi\delta(\Omega-\Omega_0)e^{j\Omega n}d\Omega && (8)\\ &= e^{j\Omega_0 n} && (9)\end{aligned}$$

taking a 2π interval that contains Ω_0 and using the sifting property of the unit impulse function $\delta(\cdot)$.

Putting together this result with the Fourier Series result, as in continuous time, we get

$$x[n] \leftrightarrow 2\pi \sum_{k \in <N>} \sum_{l=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0 + 2\pi l) \quad (10)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) \quad (11)$$

by periodicity of the $a_k' s$.

With $\Omega_0 = \frac{2\pi}{N}$, this gives us Formula 12 of Table 12.1 in the textbook (with typo fixed):

$$x[n] \text{ periodic with period } N \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - \frac{2\pi k}{N})$$

where

$$a_k = \frac{1}{N} \sum_{n \in <N>} x[n] e^{-j2\pi nk/N}$$

$$= \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j2\pi nk/N}$$

are Fourier Series coefficients (you sum up over one period of the signal).

Ex. Fourier Series analysis by inspection. Find and sketch a_k for

$$x[n] = 2 + 2\cos(\frac{\pi}{2}n) + \cos(\frac{\pi}{3}n)$$

Ex. Find the DTFT of the discrete-time impulse train

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

We see that:

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \leftrightarrow \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{N}) = P(\Omega)$$

Now let's derive this result taking a different approach that will lead to different insights.

Notation: $x[n]$ is a periodic signal with period N. Let $x_0[n]$ be the part of $x[n]$ that is repeated, i.e.

$$x_0[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

We can take the DTFT of $x_0[n]$:

$$X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n]e^{-jn\Omega} = \sum_{n=0}^{N-1} x_0[n]e^{-jn\Omega}$$

Now, we can also write $x[n]$ as an infinite sum of the function $x_0[n]$ shifted N units at a time:

$$x[n] = \sum_{k=-\infty}^{\infty} x_0[n-kN] = \sum_{k=-\infty}^{\infty} x_0[n] * \delta[n-kN] = x_0[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

We get from the convolution property that its DTFT $X(\Omega)$ is:

$$x[n] = x_0[n] * p[n] \longleftrightarrow X_0(\Omega)P(\Omega) = X(\Omega)$$

then using the DTFT of the impulse train that we just found

$$X(\Omega) = X_0(\Omega)\left(\frac{2\pi}{N}\sum_k \delta(\Omega - \frac{2\pi k}{N})\right) \quad (12)$$

$$= \frac{2\pi}{N}\sum_k X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N}) \quad (13)$$

by the property of multiplication by an impulse.

Ex. Examine $X_0(\frac{2\pi k}{N})$. How many distinct values does it have?

The inverse DTFT formula is:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_0^{2\pi}[\frac{2\pi}{N}\sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N})]e^{j\Omega n}d\Omega$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N})\int_0^{2\pi}\delta(\Omega - \frac{2\pi k}{N})e^{j\Omega n}d\Omega = \frac{1}{N}\sum_{k=0}^{N-1} X_0(\frac{2\pi k}{N})e^{\frac{j2\pi kn}{N}}$$

by the sifting property and because only the impulses for k between 0 and $N-1$ occur in the range from 0 to 2π.

Therefore, if we compare to the Fourier Series formulation on page 105, we get that

$$a_k = \frac{1}{N}X_0(\frac{2\pi k}{N})$$

In summary, we have:

$$x[n] = x_0[n] * \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

$$X_0(\Omega) = \sum_{n=0}^{N-1} x_0[n] e^{-j\Omega n}$$

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N}) \delta(\Omega - \frac{2\pi k}{N})$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_0(\frac{2\pi k}{N}) e^{\frac{j2\pi kn}{N}}$$

$$a_k = \frac{1}{N} X_0(\frac{2\pi k}{N})$$

The procedure to calculate a DTFT of a periodic DT signal is as follows:

1. Start with $x_0[n]$ and N.
2. Find $X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n] e^{-j\Omega n}$
3. Obtain $X_0(\Omega)$ at $\Omega = \frac{2\pi k}{N}, k = 0, 1, \ldots, N-1$
4. Obtain
$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N}) \delta(\Omega - \frac{2\pi k}{N})$$

Ex. $x[n] = 1$. Find $X(\Omega)$

Ex. Let $x_0[n] = \delta[n] + \delta[n-1] + 2\delta[n-3]$. Assume that $N = 4$. Find $X_0(\Omega)$ and $X(\Omega)$ and determine the four distinct values of $X_0(\frac{2\pi k}{N})$.

Ex. Given a periodic signal $x[n]$ with $N = 3$ with associated $x_0[n] = \delta[n] + 2\delta[n-2]$, find $X_0(\Omega)$ and $X(\Omega)$. Check your work by taking the inverse FFT to recover $x[n]$ (do this with MATLAB).

Ex. Given a periodic signal $y[n]$ with $N = 3$ with associated $y_0[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$, find $Y_0(\Omega)$ and $Y(\Omega)$.

12.5 Discrete Fourier Transform (DFT)

This development of the DFT is slightly different from that in your text. I think it is easier to follow. Be sure to read both of them.

Usually, we do not have an infinite amount of data which is required by the DTFT. Instead, we have 1 image, a segment of speech, etc. Also, most real- world data are not of the convenient form $a^n u[n]$.

Finally, on a computer, we can not calculate an uncountably infinite (continuum) of frequencies as required by the DTFT.

ACTUAL DATA ANALYSIS on a computer- Use a DFT to look at a finite segment of data.

In our development in the previous section of $x[n]$ periodic with $x_0[n]$ the part of the signal that was repeated, we could have assumed that our finite segment of data came from "windowing" an infinite length sequence $x[n]$

$$x_0[n] = x[n]w_R[n]$$

where $w_R[n]$ is a rectangular window:

$$w_R[n] = \begin{cases} 1, & n = 0, 1, \cdots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

$x_0[n] = x[n]w_R[n]$ is just the samples of $x[n]$ between $n = 0$ and $n = N-1$. $x_0[n]$ is 0 everywhere else. Therefore, it is defined $\forall n$, and we can take its DTFT:

$$X_0(\Omega) = \text{DTFT}(x_0[n]) = \sum_{n=-\infty}^{\infty} x_0[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]w_R[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

So,

$$X_0(\Omega) = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x_0[n]e^{-j\Omega n}$$

as we saw before.

Let's say now that we want to sample $X_0(\Omega)$ (which is continuous and periodic with period 2π) so we store it on a computer.

Sample $X_0(\Omega)$:

Assume we want 8 points in frequency – then sample $X_0(\Omega)$ at 8 uniformly spaced points on the unit circle:

Values of frequency are $0, \pi/4, \pi/2, \cdots, 7\pi/4$ *or* $2\pi k/8, k = 0, 1, \cdots, 7$.

If we let $k = N$, what happens? If $k = N$, we get repetition of the points we sampled so only N samples are unique.

Define Discrete Fourier Transform (DFT) as

$$X[k] = X_0(\frac{2\pi k}{N})$$

for $\Omega = \frac{2\pi k}{N}, k = 0, 1, \ldots, N-1$, i.e. only look at the N distinct sampled frequencies of $X_0(\Omega)$.

Note: The *resolution* of the samples of the frequency spectrum is $\frac{2\pi}{N}$ since we sample the spectrum at points that are spaced $\frac{2\pi}{N}$ apart in frequency, that is, $\Delta\Omega = \frac{2\pi}{N}$.

Note: If we looked at the samples of $X_0(\Omega)$ for all $k = -\infty$ to ∞ for frequencies $\frac{2\pi k}{N}$, we would get the closely related Discrete Fourier Series (DFS) which is of course periodic with period N since $X_0(\Omega)$ is periodic.

The DFT is just one period of the DFS.

DFT is more useful because who wants to store all the samples of a periodic signal?!! The DFT is just one period of the DFS.

Now

$$\begin{aligned} X[k] &= X_0(\Omega)|\Omega = \frac{2\pi k}{N},\ k = 0, 1, \cdots, N-1 \\ &= \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}|_{\Omega=\frac{2\pi k}{N}, k=0,1,\cdots,N-1} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}, k = 0, 1, \cdots, N-1 \end{aligned}$$

Shorthand Notation for the DFT:

Let $W_N = e^{-j\frac{2\pi}{N}} \Rightarrow N^{th}$ root of unity $(W_N^N = 1)$ since $W_N^N = e^{-j2\pi} = 1$. You may also write W_N simply as W.

Then

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, k = 0, 1, \cdots, N-1$$

is the DFT of your windowed sequence $x_0[n]$.

Ex.
Find

$$\sum_{n=0}^{N-1}(e^{\frac{-j2\pi k}{N}})^n = \sum_{n=0}^{N-1} W^{kn}$$

This is just the DFT of $x[n] = 1, n = 0, 1, \cdots, 7$.

Ex.

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & n = 1, \ldots, 7 \end{cases}$$

Find $X[k],\ k = 0, 1, \ldots, 7.$

Given $y[n] = \delta[n-2]$ and $N = 8$, find $Y[k]$.

Ex. $x[n] = cW_N^{-pn}, n = 0, 1, \ldots, N-1$,

p is an integer with $p \in [0, 1, \ldots, N-1]$ and $W_N = e^{-j\frac{2\pi}{N}}$ (as usual), find its DFT.

Synthesis: INVERSE DFT

How can we recover $x[n]$ from $X[k]$?

Synthesis formula is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; n = 0, 1, \ldots, N-1$$

Prove this gives back $x[n]$:

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} x[l] W_N^{kl} W_N^{-kn} \\ &= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} W_N^{k(l-n)} \end{aligned}$$

Ex.

$$\sum_{k=0}^{N-1} W_N^{k(l-n)} = ?$$

ORTHOGONALITY OF EXPONENTIALS AGAIN!
So,

$$\sum_{k=0}^{N-1} W_N^{k(l-n)} = \begin{cases} N, & l = n \\ 0, & l \neq n \end{cases}$$

and

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} W_N^{k(l-n)} = \frac{1}{N} \sum_{l=0}^{N-1} x[l] N \delta[n-l] \\ &= \frac{1}{N}(Nx[n]) = x[n] \end{aligned}$$

Ex. Find the IDFT of $X[k] = 1, k = 0, 1, \ldots, 7$.

Ex. Given $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + \delta[n-3]$ and $N = 4$, find $X[k]$.

Ex. Given $X[k] = 2\delta[k] + 2\delta[k-2]$ and $N = 4$, find $x[n]$.

12.6 Fast Fourier Transform

The work of Cooley and Tukey showed how to calculate the DFT with complexity $N \log N$ (called the Fast Fourier Transform) instead of complexity N^2 using the direct algorithm. The fft command that you use in MATLAB implements a Fast Fourier Transform.

Examine:

$$X[k] = \sum_{n=0}^{N-1} x[n] W^{kn}$$

There are approximately N^2 complex multiplications and additions required to implement this (N for each value of $X[k]$).

If $N = 2^{10} = 1024$, then $N^2 = 2^{20} = 10^6$, a very large number!

However, the FFT would only require about 5000, a substantial savings in complexity (the actual calculation is $\frac{N}{2} \log_2 N$).

There are a number of different FFT algorithms that exist including decimation-in-time and decimation-in-frequency.

The primary idea is to split up the size-N DFT into $\frac{N}{2}$ DFTs of length 2 each.

You split the sum into 2 subsequences of length $\frac{N}{2}$ and continue all the way down until you have $\frac{N}{2}$ subsequences of length 2.

First break $x[n]$ into even and odd subsequences:

$$X[k] = \sum_{n \text{even}} x[n]W^{kn} + \sum_{n \text{odd}} x[n]W^{kn}$$

Now let $n = 2m$ for even numbers and $n = 2m + 1$ for odd numbers:

$$X[k] = \sum_{m=0}^{\frac{N}{2}-1} x[2m]W^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1]W^{k(2m+1)} =$$

$$\sum_{m=0}^{\frac{N}{2}-1} x[2m](W^2)^{mk} + W^k \sum_{m=0}^{\frac{N}{2}-1} x[2m+1](W^2)^{mk} =$$

$$X_e[k] + W^k X_o[k]$$

$X_e[k]$ and $X_o[k]$ are both the DFT of a $\frac{N}{2}$ point sequence.

W^k is often referred to as the "twiddle factor."

Now break up the size $\frac{N}{2}$ subsequences in half by letting $m = 2p$:

$$X_e[k] = \sum_{p=0}^{\frac{N}{4}-1} x[4p](W^4)^{kp} + W^{2k} \sum_{p=0}^{\frac{N}{4}-1} x[4p+2](W^4)^{kp} =$$

The first subsequence here is the terms $x[0], x[4], \ldots$ and the second is $x[2], x[6], \ldots$.

Also, we have that:

$$W_N^{\frac{N}{2}} = -1$$

$$Y[k] = \sum_{n=0}^{1} y[n]W_2^{kn} = y[0] + W_2^k y[1]$$

$$W_2 = e^{-\frac{j2\pi}{2}} = e^{-j\pi} = -1$$

So we get,

$$Y[k] = y[0] + (-1)^k y[1]$$

and:

$$Y[0] = y[0] + y[1]$$

$$Y[1] = y[0] - y[1]$$

Ex.

This was a problem I had on a DSP final exam in 1984:

Express the DFT of the 9-point sequence $\{x[0], x[1], \ldots, x[8]\}$ in terms of the DFTs of 3-point sequences $\{x[0], x[3], x[6]\}$, $\{x[1], x[4], x[7]\}$, and $\{x[2], x[5], x[8]\}$

We start with:

$$X[k] = \sum_{m=0}^{2} x[3m] W_9^{3mk} + \sum_{m=0}^{2} x[3m+1] W_9^{(3m+1)k} + \sum_{m=0}^{2} x[3m+2] W_9^{(3m+2)k}$$

12.7 Applications of the DFT

Linear Convolution

You can use an FFT in MATLAB to compute linear convolution. If you don't use a sufficient number of points in the DFT, you will get overlap.

CIRCULAR CONVOLUTION

Since DFTs are a limited length sequence, convolution is done mod $N \Rightarrow$ circular convolution.

$$x_1[n] \otimes x_2[n] = \sum_{p=0}^{N-1} x_1[p] x_2[n-p]_{mod\ N}$$

That is, when we flip and shift the sequence $x_2[n]$, we do it mod N.

Also, note that:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1[k] X_2[k]$$

Ex. Find $x_1[n] \otimes x_2[n] = z[n]$.

Ex. $N = 8$, find $x_1[n] \otimes x_2[n] =$ y[n].

Ex. Find $y[n] = x[n] \otimes x[n]$.

More on Sampling

In this unit, we will discuss how to get Hertz on the x-axis of your Matlab plots instead of Ω.

Given two continuous time signals $x_1(t)$ and $x_2(t)$, let's see what happens if we sample them with different frequencies.

We let $\Omega = \omega T_s$ where T_s is the sampling period ($T_s = \frac{1}{f_s}$).

So, to get frequency in Hertz,

$$2\pi f = \omega = \frac{\Omega}{T_s} \Rightarrow$$

$$f = \frac{\Omega}{2\pi T_s} = \frac{\Omega f_s}{2\pi}$$

OR

$$f = \frac{\Omega}{2\pi} f_s$$

This shows us that the frequency of a reconstructed signal is related to the sampling frequency f_s by a fraction of $\frac{\Omega}{2\pi}$ of f_s.

Ex.

You sample a 40 Hz. sinusoid at a sampling frequency of 60 Hz. You reconstruct a sinusoid from its samples. What frequency will the reconstructed sinusoid have?

Ex.

You sample a 40 Hz. sinusoid at 120 Hz. What frequency will the reconstructed sinusoid have?

Ex.

You sample a 30 Hz. sinusoid at a sampling frequency of 40 Hz. You reconstruct a sinusoid from its samples. What frequency will the reconstructed sinusoid have?

Ex.

You sample a 149 Hz. sinusoid at 150 Hz. What frequency will the reconstructed sinusoid have?

Review of Chapter 12 – Fourier Analysis for Discrete-Time Signals

1. DTFT for infinite length sequences:

   continuous frequency, periodic with period 2π,

$$X(\Omega) = \sum_n x[n] e^{-j\Omega n}$$

   Important: DT convolution $\leftrightarrow$ multiplication of DTFTs.

   Inverse system:

$$h[n] * h_i[n] = \delta[n] \leftrightarrow H(\Omega) H_i(\Omega) = 1$$

2. DFT for a finite length data

   discrete frequency

   Take DTFT of windowed infinite length sequence and then sample at discrete frequencies,

$$\Omega = \frac{2\pi k}{N}, \ k = 0, 1, \ldots, N-1$$

   N discrete frequencies since exponential is periodic.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

   Orthogonality of Exponentials:

$$\sum_{n=0}^{N-1} W_N^{kn} = N\delta[k]$$