EE 235 Lecture Notes

Signal Analysis – First course in Signal Processing, Communications, Controls.

- Signal: a set of information or data that can be modeled as a function of one or more independent variables (e.g. $t \in \Re$). Examples – Speech, image, weather information, sales information, voltage in a circuit, car locations on the freeway, video, music, etc.
- System: modifies signals or extracts information that is described by input-output relations (equations). Also can be considered a transformation that operates on a signal. Examples: electronics, radio or TV, MR system, guidance system, communication system, filter, equalizer, synthesizer, etc.

Signal is input, output or internal functions that are processed or produced by system.

Examples of Signals and Systems

1. Human vocal tract (system) has air as an input signal and produces speech as an output signal
2. Voltage (signal) in an RLC circuit (system)
3. Music (signal) produced by a musical instrument (system)
4. Radio (system) has radio frequency as input and the output is the sound you listen to

Continuous Time – data are available or measured every time (or space) instant – the data are defined over a continuous input.

Discrete Time – defined only for discrete points in time such as every minute, hour, 3 months, year, etc. You sample or collect data at these times only.

This course covers continuous-time signals and rules of operations. You will cover discrete time signals in EE341.

Examples of Signal Processing

1. Spectrum Analysis – Determine cyclic patterns or other information in a signal and display it – spectrogram
2. Filter – change signal in some way – smooth, reduce noise, remove the signal in a certain frequency band such as a low pass filter
3. Prediction – predict the future based on past trends – sales, stock market, inventory
4. Synthesizer – generate output signal – speech, music, signal generator.
5. Communication channel– convey a signal from one location to another.

Speech coding (an example of a communications or signal processing application) – process speech and transmit a likeness to the original speech signal. The steps are:

1. Determine filter parameters
2. Determine pitch and energy
3. Quantize and send
4. Synthesizer generates speech at receiver.

Other examples of signal processing: CD players, Caruso recordings, image and video compression, CT, magnetic resonance imaging, modems, speech enhancement, MP3 encoding, coding a signal for wireless transmission, etc.

Chapter 2 – Continuous-Time Signals and Systems

2.1 Transformations of Continuous-Time Signals

Continuous time signal – time is a continuous variable–the signal itself need not be continuous.

Time Reversal

$$y(t) = x(-t)$$

Ex. Find $y(t) = x(-t)$

Time Scaling

$y(t) = x(at)$ SPEED UP ($|a| > 1$) or SLOW DOWN ($|a| < 1$) by a factor of a

Ex. $y(t) = x(2t)$ THIS SPEEDS UP $x(t)$

What happens to the period T?

Ex. $z(t) = x(t/2)$, SLOWS DOWN $x(t)$

Try using a table to get the main break points:

Ex. Given $y(t) = 2u(t) - 2u(t-3)$, find $y(3t)$ and $y(\frac{t}{3})$.

Time Shifting

$$y(t) = x(t - t_0)$$

Here, $y(t)$ is a version of the original signal $x(t)$ that has been shifted by an amount t_0.

Rule: set argument $t - t_0 = 0$ and move origin of $x(t)$ to t_0.

Ex. Given $x(t) = u(t+2) - u(t-2)$, find $x(t - t_0)$ for $t_0 > 0$ and $t_0 < 0$.

Ex. Determine $[x(t) + x(2-t)]$where $x(t) = u(t+1) - u(t-2)$

Method I to find $x(2-t)$: Reverse in time, then delay.
Let $y(t) = x(-t)$

Then $y(t-2) = x(-(t-2)) = x(2-t)$

Method II to find $x(2-t)$: Advance, then reverse in time
$x(2-t) = x(-t+2)$

$v(t) = x(t+2)$

$v(-t) = x(-t+2)$.

Remember: When time-shifting, move the origin of the function to the value of t such that the argument of the shifted function $= 0$.

Combinations of Scale and Shift Find $x(2t+1)$ where $x(t)$ is:

Method 1: Shift then scale: $x(at+b)$: (i) $v(t) = x(t+b)$; (ii) $y(t) = v(at) = x(at+b)$.

$$\begin{aligned} v(t) &= x(t+1) \\ y(t) &= v(2t) \end{aligned}$$

Method 2: Scale then shift: $x(at+b) = x(a(t+\frac{b}{a}))$: (i) $w(t) = x(at)$; (ii) $y(t) = w(t+\frac{b}{a}) = x(at+b)$.

$$\begin{aligned} w(t) &= x(2t) \\ y(t) &= w(t+1/2) = x(2(t+1/2)) = x(2t+1) \end{aligned}$$

Amplitude Operations

$$y(t) = ax(t) + b$$

Example: Given $x_1(t)$, find $-x_1(t), 2x_1(t)$, and $.5x_1(t)$

Example: Given $x_2(t)$, find $1 - x_2(t)$. This is a case of adding together two signals.

Addition of two signals: $x_1(t) + x_2(t)$

Multiplication of two signals: $x_1(t)x_2(t)$

2.2 Signal Characteristics

Even and Odd Functions

Any continuous time signal can be expressed as the sum of an even signal and an odd signal:

$$x(t) = x_e(t) + x_o(t)$$

$$\begin{array}{rrcl} \text{Even:} & x_e(t) & = & x_e(-t) \\ \text{Odd:} & x_o(t) & = & -x_o(-t) \end{array}$$

$$\begin{aligned} x_e(t) &= \frac{1}{2}(x(t) + x(-t)) \\ x_o(t) &= \frac{1}{2}(x(t) - x(-t)) \end{aligned}$$

Example, given the *unit step function* (a discontinuous continuous-time signal),

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{undefined}, & t = 0 \end{cases}$$

find $u_e(t)$ and $u_o(t)$

Periodic Functions

How do we tell if a continuous-time signal $x(t)$ is periodic? That is, given t and T, is there some period $T > 0$ such that

$$x(t) = x(t + T)?$$

If $x(t)$ is periodic with period T, it is also periodic with period nT, that is:

$$x(t) = x(t + nT)$$

The minimum value of T that satisfies $x(t) = x(t + T)$ is called the *fundamental period* of the signal and we denote it as T_0.

The *fundamental frequency* of the signal in hertz (cycles/second) is

$$f_0 = \frac{1}{T_0}$$

and in radians/second, it is

$$\omega_0 = \frac{2\pi}{T_0}$$

If $x_1(t)$ is periodic with period T_1 and $x_2(t)$ is periodic with period T_2, then the sum of the two signals $x_1(t) + x_2(t)$ is periodic with period equal to the least common multiple(T_1, T_2) if the ratio of the two periods is a rational number, i.e.:

$$\frac{T_1}{T_2} = \frac{k_2}{k_1}$$

Let $T' = k_1 T_1 = k_2 T_2$, and $z(t) = x_1(t) + x_2(t)$,

$$z(t + T') = x_1(t + k_1 T_1) + x_2(t + k_2 T_2) = x_1(t) + x_2(t) = z(t).$$

Examples of periodic signals are infinite sine and cosine waves.

2.3 Common Signals in Engineering

Exponential Signals

$$x(t) = Ce^{at}, \quad C \text{ and } a \text{ can be complex}$$

$a = \sigma + j\omega_0$, $\sigma =$ real part, $\omega_0 =$ imaginary part.

These are very important because complex exponentials are "eigenfunctions" of LTI systems:

As we will see, if we input $x(t) = e^{at}$, we will get $y(t) = H(a)e^{at}$ as output.

*** EULER's FORMULA *** – Memorize:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

1. Case 1, C and a real, $x(t) = Ce^{at}$

   - $a = \sigma > 0$.
     GROWING exponential,
     Chemical reactions, uninhibited growth of bacteria (potato salad), human population?

   - $a = \sigma < 0$
     DECAYING exponential
     Radioactive decay, RC circuit response, damped ME system.

   - $a = \sigma = 0 \Rightarrow x(t)$ is constant (DC) signal.

2. Case 2, C complex, a imaginary

   a is purely imaginary ($\sigma = 0$), $a = j\omega_0$

   $C = Ae^{j\phi}$ where A and ϕ are real

   $x(t) = Ce^{at} = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\phi+\omega_0 t)} = A\cos(\phi+\omega_0 t) + jA\sin(\phi+\omega_0 t)$

   $x(t)$ is a complex sinusoid.

   If C is real ($\phi = 0$), then $x(t) = A\cos\omega_0 t + Aj\sin\omega_0 t$.

   In both cases, $x(t)$ is periodic, i.e.

$$x(t) = x(t+T) \quad \text{where } T \text{ is the period}$$

   Why is $x(t)$ periodic?

$$\begin{aligned} x(t) &= Ce^{j\omega_0 t} \\ x(t+T) &= Ce^{j\omega_0 (t+T)} = Ce^{j\omega_0 t}e^{j\omega_0 T} = Ce^{j\omega_0 t} \end{aligned}$$

   Now $T = \frac{1}{|f_0|}$ and $\omega_0 = 2\pi f_0$ so $T = \frac{2\pi}{|\omega_0|}$

   What does $e^{j\omega_0 T} =$?

A real sinusoid:

T is the fundamental period

3. Case 3

   In the most general case, C and a are complex:

$$\begin{aligned} C &= Ae^{j\phi} \\ x(t) &= Ce^{at} = Ae^{j\phi}e^{at} = Ae^{j\phi}e^{(\sigma+j\omega_0)t} \\ &= Ae^{j(\phi+\omega_0 t)}e^{\sigma t} \quad \text{where } e^{\sigma t} \text{ is the damping factor} \\ &= Ae^{\sigma t}\left[\cos(\omega_0 t+\phi)+j\sin(\omega_0 t+\phi)\right] \end{aligned} \tag{1}$$

   The real part is $Ae^{\sigma t}\cos(\omega_0 t+\phi)$

   The imaginary part is $Ae^{\sigma t}\sin(\omega_0 t+\phi)$

Example: Plot real part of this signal for $\sigma > 0$ and $\sigma < 0$

2.4 Singularity Functions

Unit Step Function

We already defined the unit step function $u(t)$ as:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{undefined}, & t = 0 \end{cases}$$

Ex: Find and plot $u(t - t_0)$

Ex. Define a block function (window) as

$$rect(\frac{t}{T}) = \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Then $rect(\frac{t}{T}) = u(t + \frac{T}{2}) - u(t - \frac{T}{2})$ Plot $rect(\frac{t}{T})$:

Unit Impulse Function

Now let's look at a signal $\tilde{u}(t)$:

What is its derivative? Define it as:

$$\tilde{\delta}(t) = \frac{d}{dt}\tilde{u}(t)$$

which has unit area.

Now, $\lim_{\Delta \to 0} \tilde{u}(t) = u(t)$.

So what if we take $\lim_{\Delta \to 0} \tilde{\delta}(t)$?

The pulse height gets higher and higher and its width goes to 0, but its area is still 1!

So define $\delta(t)$ as unit impulse:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases}$$

AND

$$\int_{-\infty}^{\infty} \delta(t) = 1$$

or equivalently,

$$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \text{undefined}, & t = t_0 \end{cases}$$

AND

$$\int_{-\infty}^{\infty} \delta(t - t_0) = 1$$

Also,

$$\int_{t_1}^{t_2} \delta(t)dt = \begin{cases} 1, & \text{if } t_1 < 0 < t_2 \\ 0, & \text{otherwise} \end{cases}$$

$\delta(t)$ can be considered to be the derivative of $u(t)$ but only in a restricted sense since $u(t)$ is a discontinuous function.

$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ is a running integral

Note that the impulse function is not a true function since it is not defined for all values of t. It's a "generalized function." But its idealization will allow us to derive many interesting results.

Unit Impulse Properties

1. Scaling

   $K\delta(t)$ is an impulse with weight or area K:

2. Multiplication of a function $x(t)$ (that is continuous at 0) by an impulse $\delta(t)$:

   We get an impulse with area or weight $x(0)$.

3. Time Shift of an impulse

$$y(t) = x(t)\delta(t - t_0)$$

$$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \text{undefined}, & t = t_0 \end{cases}$$

So we get an impulse with weight equal to the value of $x(t)$ where the impulse is located:

$$y(t) = x(t_0)\delta(t - t_0)$$

Example What is $Kx(t)\delta(t - t_0)$?

Example What is $3u(t - 1)\delta(t)$?

SIFTING PROPERTY

What if we multiply a function by an impulse and then integrate?

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = ? \\ = & \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt \\ = & \; x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0) \end{aligned}$$

We integrate out the time variable so the integral is just equal to a number (or later on, a function). We'll see this many times this quarter. In this case, the impulse $\delta(t-t_0)$ is defined by the integral (as long as the function $x(t)$ is continuous at t_0).

Ex. What is

$$\int_{-\infty}^{\infty} \delta(t-a)\sin^2(\frac{t}{b})dt?$$

ALSO:

- $\int_{-\infty}^{\infty} f(t-t_0)\delta(t-t_1)dt = f(t_1-t_0)$, if $f(t)$ is continuous at t_1-t_0
- $\int_{-\infty}^{t} \delta(\tau-t_0)d\tau = u(t-t_0)$
- $\delta(t) = \delta(-t)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$

2.6 Continuous-Time Systems

A SYSTEM is an operation for which cause-and-effect relations exist.

2.7 Properties of Continuous-Time Systems

Here, we discuss some of the properties that a continuous-time system could have. We will use $x(t)$ for the input to the system, $y(t)$ as its output, and use the notation:

$$y(t) = T[x(t)]$$

or

$$y(t) = S[x(t)]$$

or

$$x(t) \rightarrow y(t)$$

Systems with memory

Systems whose output $y(t_0)$ depends on values of the input other than just $x(t_0)$ have memory.

A system $y(t_0)$ has memory if its output at time t_0 depends on the input $x(t)$ for $t > t_0$ or $t < t_0$, i.e. it depends on values of the input other than $x(t_0)$.

Otherwise, the system is MEMORYLESS

Example of a memoryless system: Resistor $v(t_0) = Ri(t_0)$; the voltage depends only on current at time t_0.

Example of System with Memory: Capacitor $v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt$; the voltage depends on past values of current so a capacitor has memory.

Ex. Given $y(t) = x(t) + 5$ and $z(t) = x(t + 5)$, which has memory?

Ex. Given $y(t) = (t+5)x(t)$ and $z(t) = [x(t+5)]^2$, do these have memory? How about $a(t) = x(5)$?

Inverse of a System

A system is invertible if you can determine the input uniquely from the output, i.e. there is a one-to-one relationship between the input and output.

Resistor is Invertible, $x(t) = i(t)$, $y(t) = v(t)$, $x(t) = y(t)/R$.
$y(t) = x^5(t)$ is an invertible system.

Noninvertible:

$$
\begin{aligned}
y(t) &= x(t)u(t) \ \Rightarrow \ \text{zeros out much of the input} \\
y(t) &= x^2(t) \ \Rightarrow \ \text{don't know sign} \\
y(t) &= \cos[x(t)] \ \Rightarrow \ \text{add } 2\pi \text{ to } x(t)
\end{aligned}
$$

Causality

Output $y(t)$ depends only on past and present inputs and **not on the future.**

All physical real-time systems are causal because we can not anticipate the future.

Image processing–Non-causal filters like blurring masks.

Music processing – record and process later – noncausal but not real-time

Ex. Resistor and Capacitor are causal,

- $v(t_0) = i(t_0)R \rightarrow$ memoryless $\Rightarrow$ Causal
- $v(t_0) = \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt \rightarrow$ Causal since only depends on past and present

Ex. $y(t_0) = \int_{-\infty}^{t_0+a} x(t)dt$.
Is this Causal? You fill in.

FACT: Memoryless $\Rightarrow$ Causal but not vice versa. In fact, most causal systems have memory.

Ex. Let $y(t) = x(-t)$.
Is this causal? Try letting t be a negative number.

Stability

Bounded Input - Bounded Output (BIBO) Stability

Input $x(t)$ bounded produces bounded output.

If $|x(t)| \leq B_1 \ \Rightarrow |y(t)| \leq B_2$, where $y(t)$ is output.

Example: Resistor is stable $V = iR$,

$$|i(t)| \leq B_1 \Rightarrow |v(t)| \leq RB_1$$

Example: Capacitor: $i(t) = C\frac{dV_c(t)}{dt}$.

Is this stable?

Let $i(t) = B_1 u(t), \text{ where } B_1 \neq 0$

$V_c(t) = \frac{B_1 t}{C}$ grows linearly with t and as $t \to \infty$, $V_c(t) \to \infty$. So capacitor is not BIBO stable.

Time-Invariance

If you shift your input signal in time for a time-invariant system, all that will happen is you will get a similar shift in your output signal. Alternatively, the system behaves the same each day and does not change over time.

If $y(t - t_0) = S[x(t - t_0)]$, then system is Time-Invariant. Else it is Time-Varying.

Ex. Is a capacitor time-invariant?

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

compare $v(t-t_0)$ with $S[i(t-t_0)]$:

Ex. Resistor $v(t) = i(t)R$. Is this Time-Invariant?

Ex.

$$y(t) = tx(t)$$

Is this Time-Invariant?

Ex. Time Reversal

$$y(t) = x(-t)$$

Is this Time-Invariant?

Also, show graphically with $x(t) = u(t) - u(t-2)$ and a delay of 1.

Ex. Test the following systems for time-invariance:

1.

$$z(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

2.

$$y(t) = \int_{0}^{t} x(\tau)d\tau$$

3.

$$a(t) = \sin[x(t)]$$

4.

$$b(t) = \sin(t)x(t)$$

5.

$$w(t) = x(2t)$$

6.

$$v(t) = \int_{0}^{T} x(t-\tau)d\tau$$

Linearity

For a system to be linear, it must satisfy additivity and homogeneity properties.

1. Additivity

$$S[x_1(t)] = y_1(t) \text{ and } S[x_2(t)] = y_2(t) \;\Rightarrow S[x_1(t)+x_2(t)] = y_1(t)+y_2(t)$$

2. Homogeneity or Scaling

$$S[x(t)] = y(t) \;\Rightarrow\; S[ax(t)] = ay(t)$$

Combine Additivity and Homogeneity to get SUPERPOSITION CONDITION;

$$\begin{aligned}
\text{If } S[x_1(t)] &= y_1(t) \quad \text{and } S[x_2(t)] = y_2(t) \\
\text{then } S[ax_1(t) + bx_2(t)] &= ay_1(t) + by_2(t)
\end{aligned}$$

Examples of Linear systems

Multiplication by a constant,

$$S[x(t)] = cx(t)$$

Try: $S[ax_1(t) + bx_2(t)]$.

$$\begin{aligned}
S[x_1(t)] &= cx_1(t) = y_1(t) \\
S[x_2(t)] &= cx_2(t) = y_2(t) \\
S[ax_1(t) + bx_2(t)] &= acx_1(t) + bcx_2(t) \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Therefore, linearly combined input produces linearly combined output.

Examples of Nonlinear systems

1. Squaring

$$S[x(t)] = y(t) = x^2(t)$$

   Violates homogeneity,

$$\begin{aligned} S[x(t)] &= x^2(t) \\ S[ax(t)] &= a^2x^2(t) \neq ax^2(t) \end{aligned}$$

   Also violates Additivity due to cross-terms.

2. Affine or Incrementally linear system

$$S[x(t)] = y(t) = x(t) + a$$

   Violates homogeneity,

$$\begin{aligned} S[x(t)] &= x(t) + a \\ S[cx(t)] &= cx(t) + a \neq c[x(t) + a] \end{aligned}$$

Violates Additivity,

$$
\begin{aligned}
S[x_1(t)] &= x_1(t) + a \\
S[x_2(t)] &= x_2(t) + a \\
S[x_1(t) + x_2(t)] &= x_1(t) + x_2(t) + a \neq S[x_1(t)] + S[x_2(t)]
\end{aligned}
$$

But can think of this as:

"Incrementally linear" or Affine.

Note: For a linear system, a zero input always produces a zero output.
Use Scaling property (homogeneity)
For a linear system

$$S[x(t)] = y(t) \Rightarrow S[ax(t)] = ay(t)$$

Let $a = 0$, then $S[0] = 0$.
For an affine system such as

$$S[x(t)] = x(t) + 3$$

Zero input produces non-zero output, $S[0] = 3$.
Superposition:

We can generalize superposition to more than 2 functions, i.e given a set of inputs $x_k(t)$ with a set of corresponding outputs $y_k(t)$, we can take a linear combination of any number of the inputs and get the same linear combination of corresponding outputs:

$$
\begin{aligned}
x(t) &= \sum_k a_k x_k(t) \quad \text{produces output} \\
y(t) &= \sum_k a_k y_k(t)
\end{aligned}
$$

You will find this very useful in doing some convolutions.

Are the following systems linear?

1.

$$y(t) = tx(2t)$$

2.

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

3.

$$y(t) = cos(x(t))$$

4.

$$y(t) = \begin{cases} x(t) & t < 0 \\ -x(t) & t \geq 0 \end{cases}$$

5.

$$y(t) = |x(t)|$$

Chapter 3 – Continuous-Time Linear Time-Invariant Systems

In this chapter, we will discuss linear time-invariant (LTI) systems – these are systems that are both linear and time-invariant. We will see that an LTI system has an input-output relationship described by a convolution.

3.1 Impulse Representation of Continuous-Time Signals

Using the sifting property, we can write a signal $x(t)$ as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

which is writing a general signal $x(t)$ as a function of an impulse function. This expresses the input $x(t)$ as an integral (continuum sum) of shifted impulses that are weighted by weights $x(\tau)$. Another way to put this is that you can build a CT signal out of impulses.

3.2 Continuous Time Convolution

We can write:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \text{ using the sifting property}$$

This expresses the input $x(t)$ as an integral (continuum sum) of shifted impulses that are weighted by weights $x(\tau)$.

Now take a system and define the impulse response of the system as

$$h(t) = S[\delta(t)]$$

and the response of the system to a shifted impulse as:

$$h(t, \tau) = S[\delta(t - \tau)]$$

If the system is linear, then

$$S[\alpha x_1(t) + \beta x_2(t)] = \alpha y_1(t) + \beta y_2(t)$$

Let

$$\begin{aligned} y(t) = S[x(t)] &= S[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau] \\ &= \int_{-\infty}^{\infty} x(\tau) S[\delta(t-\tau)] d\tau \text{ due to linearity} \\ &= \int_{-\infty}^{\infty} x(\tau) h(t, \tau) d\tau \end{aligned}$$

But what if the system is also Time-Invariant?
Then $S[\delta(t - \tau)] = h(t - \tau)$, since we had $S[\delta(t)] = h(t)$. Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = x(t) * h(t)$$

We have seen that if we have a linear time-invariant system, then the output is the input convolved with the system's impulse response $h(t)$. In other words, we can completely characterize an LTI system by its impulse response.

This is a very important result!

Convolution Integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Here, $h(\tau)$ is flipped and shifted across $x(\tau)$.

Convolution is a tough concept to get at first. I have 2 rules that will greatly improve the quality of your life:

1. DRAW A PICTURE
2. FLIP THE “EASY” FUNCTION

Why can we pick which function to flip?

Because convolution is commutative:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Change variables: $\lambda = t - \tau \Rightarrow \tau = t - \lambda \;\; d\tau = -d\lambda$.

$$\begin{aligned} y(t) &= -\int_{\infty}^{-\infty} x(t-\lambda)h(\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = h(t) * x(t) \end{aligned}$$

(minus signs cancel)

Let's examine convolution formula:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

1. Flip $h(\tau)$ and shift it to form $h(t-\tau)$.

   Note: $h(t-\tau)$ is a function of τ, not t!

   t is the shift parameter.

2. Fix t and multiply $x(\tau)$ with $h(t-\tau)$ for all values of τ.

3. Integrate $x(\tau)h(t-\tau)$ over all τ to get $y(t)$ which is a single value that depends on t. Remember that τ is the integration variable and that t is treated like a constant when doing the integral.

4. Repeat for all values of t.

Fortunately, it usually falls out that there are only several regions of interest and the rest of $y(t)$ is zero.

Ex. Find $y(t) = x(t) * h(t)$.

Form $x(\tau)$ and $h(t-\tau)$ (to shift $h(-\tau)$ by t, just add t to all points) and continue from there.

When you finish notice:

1. (a) nonzero "width" of $x(t) = 3$
   (b) nonzero "width" of $h(t) = 4$
   (c) nonzero "width" of $y(t) = 7$
2. $y(t)$ is "smoother" than $x(t)$ or $h(t)$

Ex.2 This example shows why it is easier to flip the "easy" function rather than the "hard" function. Try it both ways!

Find $u(t) * e^{-t}u(t)$

1. Method 1 $\rightarrow$ Flip $u(t)$
2. Method 2 $\rightarrow$ Flip $e^{-t}u(t)$

From before:
Sifting property of impulse:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

"Sifts out value of function $x(t)$ where impulse is"

Now, what if we convolve $x(t)$ with an impulse?
Example Find $x(t) * \delta(t-t_0) = y(t)$,

$$y(t) = x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-t_0)d\tau$$

Now, $\delta(t-\tau-t_0)$ is an impulse at $\tau = t-t_0$. So, by SIFTING property, we sift out the value of $x(t)$ where the impulse is. This is just $x(t-t_0)$.

So, convolve $x(t)$ with a shifted impulse, and you get $x(t)$ shifted to where impulse is.

So

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

We'll see this a lot. One important place we'll see this is when we discuss SAMPLING or discretizing a continuous time signal.

Be sure to check out the convolution web page (there is a link from the course web page).

Example Find $h(t) * x(t)$ where $x(t) = e^t u(-t)$ and $h(t) = 2u(t) - u(t-1) - u(t-2)$.

Example Find $e^{-t}u(t) * [u(t-1) - u(t-3)]$

Example Find $u(1-t) * e^{-t}u(t-1)$

Step Response: Response of a system to a unit step function. The input to the system is simply the unit step function; i.e. $x(t) = u(t)$.

This is equivalent to simply integrating the input from the infinite past up to time t.

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

<u>Ex.</u> Let $h(t) = e^{-at}[u(t) - u(t-2)]$. Find $s(t)$, the step response to $h(t)$.

Superposition (or Divide-and-Conquer):

We can directly apply superposition to find the output of LTI systems if $x(t)$ can be expressed as a linear combination of basis functions $\Phi_k(t)$.

$\Phi_k(t)$ is some convenient set of functions, for example unit impulses, unit step functions, or exponentials.

Ex. $x(t) = \sum_k a_k \Phi_k(t)$ has output

$$y(t) = \sum_k a_k \Psi_k(t)$$

where,

$$\Psi_k(t) = S[\Phi_k(t)] = h(t) * \Phi_k(t)$$

$$\Psi_k(t) = \Phi_k(t) * h(t)$$

Ex.

Find the output of the system where
$x(t) = 2u(t) - u(t-1) - u(t-2)$ and
$h(t) = e^{-at}[u(t) - u(t-2)]$.
We saw that for this system function $h(t)$,

$$s(t) = \frac{1}{a}\left[1 - e^{-at}\right][u(t) - u(t-2)] + \frac{1}{a}\left[1 - e^{-2a}\right]u(t-2)$$

Therefore $y(t) = 2s(t) - s(t-1) - s(t-2)$, i.e. finding the output is very simple using superposition.

Continuing Superposition:

Ex. 1

If you can express input as a sum of impulses, then can express output as sum of impulse responses.

Let $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ be a periodic impulse train,

Input $p(t)$ to an LTI system with impulse response $h(t)$,

Then

$$y(t) = p(t) * h(t) = \left[\sum_{-\infty}^{+\infty} \delta(t - kT) \right] * h(t).$$

We know that a function convolved with a shifted impulse yields the function shifted to where impulse is.

Therefore,

$$y(t) = \sum_{-\infty}^{\infty} h(t - kT)$$

is simply the replicated impulse response. We'll see more impulse trains when we discuss SAMPLING.

Given this $h(t)$:

and $p(t)$ the same impulse train, we get y(t) as,

What if $h(t)$ had a width greater than $T/2$?

This is related to “aliasing” and we’ll discuss it later during our discussion of sampling (A/D conversion).

Ex. 2

Let $x(t) = \sum_k a_k e^{s_k t}$ be a linear combination of exponentials where a_k and s_k are complex.

Let $\Phi_k(t) = e^{s_k t}$ and let $\Psi_k(t) = \Phi_k(t) * h(t)$

Then

$$y(t) = \sum_k a_k \Psi_k(t)$$

and

$$\Psi_k(t) = \int_{-\infty}^{\infty} h(\tau) e^{s_k(t-\tau)} d\tau = e^{s_k t} \int_{-\infty}^{\infty} h(\tau) e^{-s_k \tau} d\tau = H(s_k) e^{s_k t}$$

where $H(s_k) = \int_{-\infty}^{\infty} h(\tau) e^{-s_k \tau} d\tau$. Then $y(t) = \sum_k a_k H(s_k) e^{s_k t}$.

Notice that we put in a linear combination of complex exponentials and we get out the complex exponentials weighted by $H(s_k)$.

We will see in Fourier Analysis that the complex exponentials are eigenfunctions of LTI systems:

$$\begin{aligned} e^{s_k t} &= \text{eigenfunction} \\ H(s_k) &= \text{eigenvalue} \end{aligned}$$

3.3 Properties of Convolution

Convolution is commutative, associative, and distributive. Keeping this in mind may simplify some convolutions for you.

- Commutative:
- Associative:
- Distributive:

Cascade Interconnections

$$w(t) = x(t) * h_1(t), \quad y(t) = w(t) * h_2(t)$$
$$= [x(t) * h_1(t)] * h_2(t)$$
$$= x(t) * [h_1(t) * h_2(t)], \text{ by associativity of convolution}$$

Therefore $h(t)$ for this overall system is $h_1(t) * h_2(t)$.

Can change order of operations due to commutativity. For a cascade of M systems there are $M!$ possible system orderings.

Parallel interconnection

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)]$$
$$\Rightarrow h(t) = h_1(t) + h_2(t)$$

Parallel systems is a large area of research today.

3.4 Properties of Continuous-Time LTI systems

We saw that I/O properties of LTI system are completely determined by system's impulse response $h(t)$ and that the output $y(t) = x(t) * h(t)$.

In this section, we will express other known system attributes in terms of conditions on $h(t)$.

Systems with memory

In a memoryless system, the output $y(t)$ is a function of the input $x(t)$ at time instant t alone.

An LTI system that is memoryless can only have this form:

$$y(t) = Kx(t)$$

K is the system gain and must be constant or else the system would vary with time.

$$y(t) = Kx(t) = x(t) * h(t)$$

For this to hold, $h(t)$ must be of the form of an impulse weighted by K:

$$h(t) = K\delta(t)$$

What if $h(t) = K\delta(t-d),\ d \neq 0$?
$y(t) = x(t) * h(t) = Kx(t-d)$.
The time shift d implies memory.
$y(t)$ depends on $x(t-d)$, not $x(t)$.

Example Is a system described by $h(t) = u(t) - u(t-1)$ memoryless?

Invertible Systems

$$h(t) * h_I(t) = \delta(t)$$

A system is invertible if we can find $h_I(t)$
We will see how to do this when we study transforms.

Causality

We know that for a causal system, the output depends only on past or present inputs and not on future inputs.

Equivalently, a causal system does not respond to an input until it occurs (the output is not based on the future)

So, a response to an input at $t = t_0$, occurs only for $t \geq t_0$.

We know that $h(t)$ is the system response to $\delta(t)$, and that $\delta(t)$ occurs at $t = 0$.

Therefore for a causal system, $h(t) = 0,\ t < 0$.

Examine convolution equation (flip $h(t)$):

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Causality: if $h(t)$ is causal then $h(t-\tau) = 0,\ t-\tau < 0$ or $t < \tau$.

So

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau$$

which shows us that the output $y(t)$ depends only on values of input $x(\tau)$ for $\tau \leq t$.

Ex.

Is $h_1(t) \quad = \quad u(t+1)$ causal?

Is $h_2(t) \quad = \quad u(t-1)$ causal?

Stability

We can tell if an LTI system is BIBO stable from its impulse response.

Given a bounded input $|x(t)| \leq B_1$, for all t, check its output to see if it remains finite:

$$
\begin{aligned}
|y(t)| = |\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau| \leq \\
\int_{-\infty}^{\infty} |x(t-\tau)h(\tau)|d\tau \quad \text{Why?} \\
= \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq \int_{-\infty}^{\infty} B_1|h(\tau)|d\tau
\end{aligned}
$$

So, $|y(t)| \leq B_1 \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ if $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$.
That is, if the impulse response $h(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau = G < \infty,$$

Then, $|y(t)| \leq B_1 G = B_2$ and system is BIBO stable (sufficient condition).

Ex. Is $h(t) = u(t)$ stable?

$$y(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau) = \int_{-\infty}^{t} x(\tau)d\tau$$

running integral of $x(t)$.

Ex. Given an impulse response $h(t) = e^{-at}u(t),\ a > 0$, is the system BIBO stable? How about for $a < 0$?

Unit Step Response

$$s(t) = h(t) * u(t)$$

$$s(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

running integral of $h(t)$.
can find $h(t)$ as derivative of $s(t)$.

$$h(t) = s'(t) = \frac{d}{dt}s(t)$$

Ex. Given a step response $s(t) = \frac{1}{a}[1-e^{-at}]u(t)$, find the system's impulse response

3.5 Differential-Equation Models

$$\frac{d}{dt}y(t) - ay(t) = bx(t)$$

is a linear first-order (the differential equation is first-order) differential equation with constant coefficients (as long as a and b are constants). Again, $x(t)$ is the system input and $y(t)$ is the output. This is an LTI system.

An important class of continuous time LTI systems are those modeled by ordinary linear differential equations with constant coefficients.

A general nth order linear differential equation with constant coefficients is:

$$a_0y(t) + a_1\frac{d}{dt}y(t) + \ldots + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + a_n\frac{d^n}{dt^n}y(t) =$$

$$b_0x(t) + b_1\frac{d}{dt}x(t) + \ldots + b_{m-1}\frac{d^{m-1}}{dt^{m-1}}x(t) + b_m\frac{d^m}{dt^m}x(t)$$

which we can write as:

$$\sum_{k=0}^{n} a_k\frac{d^k}{dt^k}y(t) = \sum_{k=0}^{m} b_k\frac{d^k}{dt^k}x(t).$$

Solution of Differential Equations

A classical method for the solution of our differential equation is called the *method of undetermined coefficients*. We express the output $y(t)$ as the sum of *complementary* or *natural* ($y_c(t)$) and *particular* or *forced* ($y_p(t)$) solutions:

$$y(t) = y_c(t) + y_p(t)$$

Natural response The natural response $y_c(t)$ is the solution to the homogeneous equation:

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = 0.$$

Assume that the solution of the homogeneous equation is:

$$y_c(t) = Ce^{st}$$

Substituting in the homogeneous equation yields:

$$(a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n) Ce^{st} = 0$$

and we get:

$$(a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n) = 0$$

This is the *characteristic equation* and it may be factored as:

$$a_n (s - s_1)(s - s_2) \ldots (s - s_n) = 0$$

The solution is of the form:

$$y_c(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \ldots + C_n e^{s_n t}$$

assuming there are no repeated roots (which is all we will cover here).

Ex. Given a first-order differential equation

$$y(t) = x(t) - .7\frac{d}{dt}y(t)$$

find its homogeneous solution. Your answer should be in terms of a constant C.

Forced response The forced response $y_p(t)$ solves the equation

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t).$$

The form of the solution is determined by the input $x(t)$. For an exponential input $x(t) = Ae^{at}$, the solution would be $y_p(t) = Pe^{at}$ where $A, a,$ and P are constants.

Ex. For the previous example, assume an input $x(t) = 6e^{3t}$. Find the particular solution $y_p(t)$.

Ex. Now, assuming that the system is initially at rest, i.e., $y(0) = 0$ meaning that the initial conditions are 0, solve for the constant C in your overall solution $y(t) = y_c(t) + y_p(t)$.

Ex. Given

$$y'(t) + 2y(t) = x(t)$$

where $x(t) = 4e^{2t}$, find $y(t)$. Assume $y(0) = 0$.

3.6 Terms in the Natural Response

Recall that the natural solution was

$$y_c(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \ldots + C_n e^{s_n t}$$

where s_i is the root of the characteristic equation

$$(a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n) = a_n (s - s_1)(s - s_2) \ldots (s - s_n) = 0$$

The root of the characteristic equation s_i can be either real or complex. If it is complex, it must occur in conjugate pairs since the coefficients of the characteristic equation are real.

- If s_i is real, then $C_i e^{s_i t}$ is exponential in form.
- If s_i is complex, then let $s_i = \sigma_i + j\omega_i$ and $C_i e^{s_i t} = C_i e^{\sigma_i t} e^{j\omega_i t}$ and the conjugate pair of these terms can be expressed as:

$$C_i e^{s_i t} + (C_i e^{s_i t})^* = 2|C_i| e^{\sigma_i t} \cos(\omega_i t + \phi_i)$$

where $C_i = |C_i| e^{j\phi_i}$.

Stability

The root s_i will determine if the overall system is BIBO stable or not.

Assume we have a *causal* LTI system. The solution is of the form

$$y(t) = y_c(t) + y_p(t)$$

where $y(t) = 0$ for $t < t_0$ (t_0 is the start time). Since $y_p(t)$ is of the form $x(t)$, if the input $x(t)$ is bounded, then $y_p(t)$ will also be bounded.

Let's examine

$$y_c(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \ldots + C_n e^{s_n t}$$

Clearly, as long as the real part of all roots (also called *poles*) of this equation, $\sigma_i < 0$ (since we've assumed that the system is causal), then each term in $y_c(t)$ will be bounded.

A necessary and sufficient condition for stability of a causal LTI system is that all roots of the system characteristic equation lie in the left half plane of the s-plane.

Ex. Given a causal LTI system described by the differential equation

$$\frac{d^2}{dt^2}y(t) - 2.5\frac{d}{dt}y(t) + y(t) = x(t)$$

determine if the system is BIBO stable.

3.7 System Response for Complex-Exponential Inputs

Given an input $x(t) = Xe^{st}$ to a BIBO stable LTI system modeled by an nth order linear differential equation with constant coefficients, we will examine the steady-state system response. Assume that X and s are complex.

The forced (steady-state) response of the system to this input is of the same form of the input, i.e.

$$y_{ss}(t) = Ye^{st}.$$

From the differential equation describing the system,

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t).$$

or,

$$a_0 y(t) + a_1 \frac{d}{dt} y(t) + \ldots + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + a_n \frac{d^n}{dt^n} y(t) =$$

$$b_0 x(t) + b_1 \frac{d}{dt} x(t) + \ldots + b_{m-1} \frac{d^{m-1}}{dt^{m-1}} x(t) + b_m \frac{d^m}{dt^m} x(t)$$

Plugging in $x(t)$ and $y_{ss}(t)$, we get

$$a_0 Ye^{st} + a_1 sYe^{st} + \ldots + a_{n-1} s^{n-1} Ye^{st} + a_n s^n Ye^{st} =$$

$$b_0 Xe^{st} + b_1 sXe^{st} + \ldots + b_{m-1} s^{m-1} Xe^{st} + b_m s^m Xe^{st}$$

which we can write as
$Y = H(s)X$
where

$$H(s) = \frac{b_0 + b_1 s + \ldots + b_m s^m}{a_0 + a_1 s + \ldots + a_n s^n}$$

is a *transfer function.*

So given an input $x(t) = Xe^{s_1 t}$ to such an LTI system, the steady-state response is $y_{ss}(t) = XH(s_1)e^{s_1 t}$.

Similar to the differential equation and the impulse response, the transfer function $H(s)$ completely characterizes the LTI system (we can derive the differential equation from $H(s)$ and vice versa).

In general, by superposition, given an input $x(t) = \sum_{k=1}^{N} X_k e^{s_k t}$, the output of the system is $y_{ss}(t) = \sum_{k=1}^{N} X_k H(s_k) e^{s_k t}$.

Eigenfunctions of CT LTI systems are complex exponentials:

$$\phi(t) = e^{st}$$

Check: What is $e^{st} * h(t)$?

$$= e^{st} * h(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau = H(s)e^{st}$$

where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$ is the eigenvalue. This motivates the Laplace transform and the Fourier Transform:

- $H(s)$ is known as the bilateral Laplace transform of $h(t)$.
- If $s = j\omega$, then we get $H(j\omega)$, the Fourier Transform of $h(t)$.

Ex.

Given an input $x(t) = e^{st}$, and

$$h(t) = e^{3t}u(t)$$

find its steady-state output

$$y_{ss}(t) = H(s)e^{st}$$

YOU FINISH:

3.8 Block Diagrams

We will not cover this section.

Chapter 4 Fourier Series

Up until now, we have only looked at signals in the time (or space) domain.

We will now transform signals to frequency domain.

Many signals are more convenient to process, analyze, synthesize, and/or compress in the frequency domain.

Ex.1 Magnetic Resonance (MR) imaging
Data are collected in frequency domain and then are inversed transformed to form the image. $s(t)$ is the signal that is collected,

$$s(t) = K \int M_0(x) e^{-[i\gamma G x + \frac{1}{T_2(x)}]t} dx$$

$M_0(x)$ is the image signal.

$s(t)$ is the Fourier Transform of $M_0(x)$.

Ex.2 Image Compression
The eye is less sensitive to errors in high spatial frequencies.

As a result, we can obtain good image quality at low bit rates using the discrete cosine transform (DCT) by carefully reproducing the low frequencies of the image while saving bits in the high frequencies. The DCT is related to the discrete Fourier transform except the exponential in the kernel is replaced by a cosine when taking the transform. The DCT is used in the JPEG and MPEG image and video compression standards. It is also used in RealNetworks' video codecs.

Ex. 3 Speech Recognition.
By displaying a speech spectrogram, researchers at MIT can determine

what the speech utterance that produced the spectrogram was. The Spectrogram is a plot of intensity of frequency vs. time.

Ex. 4 By knowing the highest frequency in a signal, we can determine the rate at which to sample the analog signal to ensure that the original signal can be recovered from the digitized samples–CD music is sampled at 44.1 kHz because music has frequencies that only go up to 20 kHz.

In this chapter on Fourier Series, we will see how to build and approximate periodic functions by a sum (possibly infinite) of sinusoidal signals.

4.1 Approximating Periodic Functions

Periodic Functions

$$x(t+T) = x(t) \Rightarrow$$

$x(t)$ is periodic with period T.

The signal $x(t) = Ae^{j\omega_0 t}$ is periodic with period $T = \frac{2\pi}{\omega_0}$.

$$x(t+T) = Ae^{j\omega_0 (t+\frac{2\pi}{\omega_0})} = Ae^{j\omega_0 t}e^{j2\pi} = Ae^{j\omega_0 t} = x(t)$$

EX. The signal

is periodic with $T = 4$.

We will see how to approximate periodic signals with complex exponentials.

4.2 Fourier Series

$x_k(t) = e^{jk\omega_0 t}$ are harmonic signals with period $T_k = \frac{2\pi}{|k|\omega_0},\ k \neq 0$.

Fourier Series idea: Represent all periodic signals as a harmonic series of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

C_k are the Fourier Series coefficients where $k \in \mathbf{Z}$.

$k = 0$ gives DC signal

$k = \pm 1$ yields the fundamental frequency or first harmonic ω_0

$|k| \geq 2$ harmonics (violin)

There are a number of different forms a Fourier Series can take. They are equivalent. Here, we will work with the Exponential form.

Ex.1 Find the Fourier Series coefficients for

$$x(t) = \cos\omega_0 t + \sin 2\omega_0 t$$

Ex. 2 Find the Fourier Series coefficients for

$$y(t) = sin^2 2\omega_0 t + 2\cos\omega_0 t = \frac{1}{2}(1 - \cos 4\omega_0 t) + 2\cos\omega_0 t$$

Ex. 3 SYNTHESIS

Given $\omega_0 = \pi, C_0 = 2, C_1 = 1, C_3 = \frac{1}{2}e^{\frac{j\pi}{4}}, C_{-3} = \frac{1}{2}e^{\frac{-j\pi}{4}}$, find the signal $y(t) = \sum_k C_k e^{jk\omega_0 t}$

Fourier Coefficients

We used Euler's formula and trigonometric identities to determine Fourier Series coefficients in our examples. Not all signals are as convenient as these were.

For general signals, we need a method to find the Fourier Series coefficients C_k:

Important: We will make use of this integral a lot (k and n are integers):

$$\underline{k \neq n:} \int_0^T e^{j(k-n)\omega_0 t} dt = \frac{1}{j(k-n)\omega_0} e^{j(k-n)\omega_0 t}\Big|_0^T = \frac{1}{j(k-n)\omega_0}\left[e^{j(k-n)\omega_0 T} - 1\right]$$

Now $T = \frac{2\pi}{\omega_0}$, so we get: $\frac{1}{j(k-n)\omega_0}\left[e^{j(k-n)2\pi} - 1\right] = 0$.
But $e^{j(k-n)2\pi} = 1$ since we just have an integer multiple of 2π

$$\underline{k = n:} \int_0^T (1) dt = T$$

Therefore, we see that,

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

which is known as the "Orthogonality of Exponentials."

We will use this to determine the Fourier Series coefficients C_k as follows:

1. Take $x(t) = \sum_k C_k e^{jk\omega_0 t}$
2. Multiply both sides by $e^{-jn\omega_0 t}$
3. Integrate over one period of the signal:

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \sum_k C_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

Switching the order of the sum and integral (this is valid in all but pathological cases),

$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_k C_k \int_T e^{j(k-n)\omega_0 t}dt$$

We've seen this integral before. We can pick any period over which to integrate our periodic signal. A convenient period is $[0, T_0)$.

$$\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt = \sum_k C_k \int_0^{T_0} e^{j(k-n)\omega_0 t}dt$$

$$\int_0^{T_0} e^{j(k-n)\omega_0 t}dt = \begin{cases} T_0, & k = n \\ 0, & k \neq n \end{cases}$$

$$\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt = C_n T_0$$

because the only term that is nonzero in $\sum_k C_k \int_0^{T_0} e^{j(k-n)\omega_0 t}dt$ is for $k = n$.

Therefore our Fourier Series coefficient $C_n = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt$.

Fourier Series Pair:

$$\begin{aligned} x(t) &= \sum_k C_k e^{jk\omega_0 t} \quad \text{synthesis equation} \\ C_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt \quad \text{analysis equation} \end{aligned}$$

for $k = 0$, $C_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$ gives the DC value which is the average value of $x(t)$ over one period.

Ex Given a signal $x(t) = \cos t + \sin 2t$, find its Fourier Series coefficients.

Ex Given a signal $y(t) = \cos 2t$, find its Fourier Series coefficients.

Ex.3 Periodic Square Wave

Given a periodic square wave, find its Fourier Series coefficients.

4.3 Fourier Series and Frequency Spectra

We can plot the *frequency spectrum* or *line spectrum* of a signal. It is a graph that shows the amplitudes and/or phases of the Fourier Series coefficients C_k. The plots are called line spectra because we indicate the values by lines.

Example: Calculate the Fourier Series coefficients for the impulse train $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$ and plot the magnitude of its frequency spectrum (which are simply the Fourier Series coefficients).

Example: Calculate the Fourier Series coefficients for the periodic square wave and plot its frequency spectrum

$$x(t) = \begin{cases} V, & 0 < t < \frac{T_0}{2} \\ -V, & \frac{T_0}{2} < t < T_0 \end{cases}$$

Notice that the spectrum for the square wave dies off as $\frac{1}{k}$ whereas for the periodic impulse train, it remains constant. Therefore, all harmonics are important for the periodic impulse train but not for the periodic square wave.

You can calculate the other Fourier Series listed in your book on page 149 for practice (Table 4.3).

4.4 Properties of Fourier Series

Skim this section in the book. The main points are about convergence of the Fourier Series.

Because Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

has an infinite number of terms in the series, we must consider if the infinite series converges. Further, we can approximate a signal $x(t)$ to any degree of accuracy with a truncated Fourier Series.

It will converge (under a mean square error norm)

1. $x(t)$ is continuous, or
2. $\int_T |x(t)|^2 dt < \infty$, (finite power) or
3. $\int_T |x(t)| dt < \infty$, (absolutely integrable over 1 period) (in most cases)

Any one of these conditions is sufficient for convergence.

4.5 System Analysis

In this section, we consider the analysis of stable LTI systems with periodic inputs that can be represented with Fourier Series. We will consider the variation of the system response to frequency, i.e. the *system frequency response.*

We saw that if we input a signal $ae^{jk\omega_0 t}$ to such an LTI system, the response is $aH(jk\omega_0)e^{jk\omega_0 t}$ (where a is a constant).

If we input a periodic input signal (expressed with its Fourier Series):

$$x(t) = \sum_k C_k e^{jk\omega_0 t}$$

to a stable LTI system, using superposition, we can write down the output:

$$y(t) = \sum_k C_k H(jk\omega_0) e^{jk\omega_0 t}$$

where

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau$$

Note: $y(t)$ is also a Fourier Series with Fourier Series coefficients $C_k H(jk\omega_0)$. The book refers to the Fourier Series coefficients of $x(t)$ as C_{kx} and those of $y(t)$ as C_{ky}. Notice that $C_{ky} = C_{kx} H(jk\omega_0)$.

Ex. Given an LTI system with impulse response $h(t) = \alpha e^{-\alpha t} u(t), \alpha > 0$. Find the output of the system to an input $x(t) = \sin^2 2t$.

HINT: First find $H(jk\omega_0)$ and then express $x(t)$ as a Fourier Series. The output will also be a Fourier Series. We will see that $h(t)$ is a LOW PASS filter. If $h(t)$ is used as a filter, it passes the LOW frequencies and cuts out the HIGH frequencies. In an image, examples of low frequencies are solid regions, while examples of high frequencies are edges.

Ex. Given the same low pass filter in the previous example and a second input signal

$$x_1(t) = 1 + \cos t + \cos 8t$$

find the output $y_1(t)$.

4.6 Fourier Series Transformations

Skim this section in the book.

Chapter 5 The Fourier Transform

This chapter will cover the Fourier Transform which is for aperiodic signals. The Fourier Transform is a method for representing signals and systems in the frequency domain.

5.1 Definition of the Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

is the continuous time Fourier transform of an aperiodic signal $f(t)$.

It is an extension of the Fourier Series. The Fourier transformation creates $F(\omega)$ in the FREQUENCY domain. We'll see that $F(\omega)$ can be seen as a "continuous coefficient" of a Fourier Series if we let the period of the periodic function go to ∞ so that the resulting function becomes aperiodic in the limit.

Let $f_p(t)$ be a periodic function with Fourier series,

$$f_p(t) = \sum_k C_k e^{jk\omega_0 t}, \quad C_k = \frac{1}{T_0}\int_{T_0} f_p(t)e^{-jk\omega_0 t}dt$$

$T_0 = \frac{2\pi}{\omega_0}$ is the period of $f_p(t)$.

Plot the spectrum of $f_p(t)$ vs. $\omega = k\omega_0$: (assuming $f_p(t)$ is a periodic rectangular pulse train),

Each C_k is the frequency component of $f_p(t)$ at the frequency $k\omega_0 = \omega$.

Now let $T_0 \to \infty$ so $f_p(t)$ is aperiodic and $\omega_0 = \frac{2\pi}{T_0} \to 0$.

Then the lines in the plot merge into a continuous spectrum.

As $\omega_0 \to 0$, the distance between lines goes to 0, so let us write $\Delta\omega$ for ω_0:

$$f_p(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} C_k e^{jk\Delta\omega t}$$

Replace formula for C_k in Fourier Series representation of $f_p(t)$

$$\begin{aligned} f_p(t) &= \sum_{k=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{T_0} f_p(\tau) e^{-jk\Delta\omega\tau} d\tau \right] e^{jk\Delta\omega t} \\ &= \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left[\int_{-\frac{\pi}{\Delta\omega}}^{\frac{\pi}{\Delta\omega}} f_p(\tau) e^{-jk\Delta\omega\tau} d\tau \right] e^{jk\Delta\omega t} \end{aligned}$$

Let $T_0 \to \infty$, ($\Delta\omega \to 0$) and define $f(t) = \lim_{T_0 \to \infty} f_p(t)$:

$$f(t) = \frac{1}{2\pi} \lim_{\Delta\omega \to 0} \sum_{k=-\infty}^{\infty} \left[\int_{-\frac{\pi}{\Delta\omega}}^{\frac{\pi}{\Delta\omega}} f(\tau) e^{-jk\Delta\omega\tau} d\tau \right] e^{jk\Delta\omega t} \Delta\omega$$

As $\Delta\omega \to 0$, we write it as $d\omega$ and the sum becomes an integral (and $k\Delta\omega$ approaches ω as $T_0 \to \infty$ where ω is a continuous variable).

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau \right] e^{j\omega t} d\omega$$

Let $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$ be the Fourier Transform of $f(t)$.
So $F(\omega)$ replaces the C_k as $\Delta\omega \to 0$ and is a continuous function of ω.

<u>Fourier transform pair:</u>

$$\begin{aligned} \text{ANALYSIS } F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ \text{SYNTHESIS } f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \end{aligned}$$

Sufficient conditions for the existence of the Fourier Transform are the Dirichlet conditions:

1. On any finite interval
   (a) $f(t)$ is bounded
   (b) $f(t)$ has a finite number of minima and maxima
   (c) $f(t)$ has a finite number of discontinuities
2. Since $|e^{-j\omega t}| = 1$, the Fourier Transform exists if $\int_{-\infty}^{\infty} |f(t)| dt < \infty$ (absolutely integrable) (except for certain pathological cases).

$$|F(\omega)| = |\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt| \leq \int_{-\infty}^{\infty} |f(t) e^{-j\omega t}| dt$$

$$\int_{-\infty}^{\infty} |f(t)||e^{-j\omega t}| dt = \int_{-\infty}^{\infty} |f(t)| dt$$

- Energy Signals:
- Power Signals:

Ex.1 Compute the Fourier Transform of

$$x(t) = \begin{cases} 1 & -T_1 \leq t \leq T_1 \\ 0 & \text{otherwise} \end{cases}$$

Note: sinc x = $\frac{sinx}{x}$ is an IMPORTANT continuous time function that we will see many times.

How to Plot a Sinc Function

Given a function $sincx = \frac{\sin x}{x}$, how do we plot it?

- The height of the sinc function at $x = 0$ is 1 because using L'Hôpital's Rule, we get $\frac{sin0}{0} = 1$.
- The zero crossings occur where $\sin x = 0$. This is when x is a multiple of π.

Ex. Plot the function $\beta sinc(\frac{\beta\omega}{2})$. It is the Fourier Transform of $rect(\frac{t}{\beta})$.

Ex.2 Compute the Fourier Transform of $\delta(t)$

Ex.3 Compute the Fourier Transform of

$$x(t) = e^{at}u(-t), a > 0$$

Ex. 4 Find $F(\omega)$ for $f(t) = C\delta(t + t_0)$

Ex.5 (Synthesis) Find $f(t)$ from

$$F(\omega) = \begin{cases} 1, & |\omega| < w_b \\ 0, & |\omega| > w_b \end{cases}$$

5.2 Properties of the Fourier Transform

The point of this Section is that knowledge of the properties of the Fourier Transform can save you a lot of work. We will cover some of those listed in the textbook here.

Linearity

The Fourier Transform is linear so

$$\mathcal{F}\left[ax_1(t) + bx_2(t)\right] = aX_1(\omega) + bX_2(\omega)$$

Time Scaling

$$x(t) \longleftrightarrow X(\omega)$$

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

$$\text{Proof: } \mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

Let $u = at$, $du = adt$, $dt = \frac{du}{a}$,

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(u)e^{-j\frac{\omega u}{a}}\frac{1}{a}du = \frac{1}{a}X(\frac{\omega}{a})$$

if $a > 0$. If $a < 0$, then

$$\mathcal{F}[x(at)] = \int_{+\infty}^{-\infty} x(u)e^{-\frac{j\omega u}{a}}\frac{du}{a} = -\frac{1}{a}X(\frac{\omega}{a})$$

(since $u = at$). Therefore

$$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

Heisenberg: Duration of a signal in time and its bandwidth in frequency are inversely proportional.

Time Shifting

$$x(t) \leftrightarrow X(\omega)$$
$$x(t-t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$$

$$\text{Proof: } \mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} x(u)e^{-j\omega(u+t_0)}du = e^{-j\omega t_0}X(\omega)$$

Duality

Note the DUALITY when you compare Example 1 and Example 5 of Section 5.1.

The Fourier Transform of a block is a sinc and the Fourier Transform of a sinc is a block.

Duality: ($\longleftrightarrow$ stands for "has Fourier Transform")

$$x(t) \longleftrightarrow X(\omega)$$
$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

Why?

$$X(\omega) = \int_{-\infty}^{\infty} x(b)e^{-j\omega b}db$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(b)e^{jbt}db$$
$$2\pi x(t) = \int_{-\infty}^{\infty} X(b)e^{jbt}db$$
$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(b)e^{-jb\omega}db$$

which is equal to

$$\mathcal{F}\left[X(t)\right] = \int_{-\infty}^{\infty} X(b)e^{-jb\omega}db$$

Ex.5 Find the Fourier Transform of

$$x(t) = e^{-at}u(t),\ \ a > 0$$

Ex.6 Find the Fourier Transform of

$$y(t) = \frac{1}{a + jt}$$

More Duality: compare

$$
\begin{aligned}
\mathcal{F}^{-1}\left[x(\omega)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(b)e^{jbt}db = \frac{1}{2\pi}X(-t) \\
X(\omega) &= \int_{-\infty}^{\infty} x(b)e^{-j\omega b}db
\end{aligned}
$$

Therefore,

$$
\begin{gathered}
x(t) \longleftrightarrow X(\omega) \\
\frac{1}{2\pi}X(-t) \longleftrightarrow x(\omega)
\end{gathered}
$$

Ex.7 Find the Fourier Transform of

$$\frac{1}{a - jt}$$

Convolution

$$x(t) * h(t) \leftrightarrow X(\omega)H(\omega)$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-j\omega t}dt$$

Now, switch order of integration (usually ok)

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau \quad \text{let } u = t-\tau \\ &= \int_{-\infty,d\tau}^{\infty} x(\tau)\int_{-\infty,du}^{\infty} h(u)e^{-j\omega(u+\tau)}dud\tau \\ &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\int_{-\infty}^{\infty} h(u)e^{-j\omega u}du \\ &= X(\omega)H(\omega) \end{aligned}$$

Therefore,

$$y(t) = x(t) * h(t) \longleftrightarrow Y(\omega) = X(\omega)H(\omega) \tag{2}$$

For LTI systems, you can avoid convolutions by using the Fourier Transforms!!

Multiplication of Signals

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

$$\begin{aligned}
x_1(t)x_2(t) \quad &\leftrightarrow \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(u)X_2(\omega-u)du \\
\text{Proof: } \mathcal{F}[x_1(t)x_2(t)] \quad &= \quad \int_{-\infty}^{\infty} x_1(t)x_2(t)e^{-j\omega t}dt
\end{aligned}$$

Now, write $x_1(t)$ as an inverse Fourier Transform.

$$\begin{aligned}
\mathcal{F}[x_1(t)x_2(t)] &= \int_{-\infty,dt}^{\infty}[\frac{1}{2\pi}\int_{-\infty,da}^{\infty} X_1(a)e^{jat}da]x_2(t)e^{-j\omega t}dt \\
&= \frac{1}{2\pi}\int_{-\infty,da}^{\infty} X_1(a)\int_{-\infty,dt}^{\infty} x_2(t)e^{-j(\omega-a)t}dtda \\
&= \frac{1}{2\pi}\int_{-\infty,da}^{\infty} X_1(a)X_2(\omega-a)da \\
&= \frac{1}{2\pi}X_1(\omega) * X_2(\omega)
\end{aligned}$$

Therefore,

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)$$

Ex.1 Find the inverse transform of

$$X(\omega) = \begin{cases} 1 & -1 \leq \omega \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Ex.2 Find the Fourier Transform of $x(t) = \text{sinc}^2 t$ (use multiplication property).

Ex.3 Find $y(t) = \text{sinc}^2 t * \text{sinc}\ t$

Frequency Shifting or Modulation

$$\begin{aligned}\mathcal{F}[e^{j\omega_0 t}x(t)] &= \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-jt(\omega-\omega_0)}dt = X(\omega-\omega_0)\end{aligned}$$

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$$

Modulation in time domain $\Longleftrightarrow$ shift in frequency domain.

5.3 Fourier Transform of Time Functions

DC Level

Ex. Find the Fourier Transform of the constant 1. Use duality.

We will see in an upcoming section that:

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$$

Fourier Transform of Periodic Signals

We can define a Fourier Transform for periodic signals. This will be more convenient to use if a signal has both periodic and aperiodic components.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

is the Fourier Series representation of the periodic signal $x(t)$.

Can we take the Fourier Transform of the periodic signal? It doesn't satisfy absolute integrability. So generalize Fourier Transform to include impulses:

$$\mathcal{F}[x(t)] = \int_{-\infty}^{\infty} \sum_k C_k e^{jk\omega_0 t} e^{-j\omega t} dt = \sum_k C_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt$$

Now,

$$\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt$$

is the Fourier Transform of $e^{jk\omega_0 t}$.

Guess (because of duality):

$$e^{jk\omega_0 t} \leftrightarrow 2\pi\delta(\omega - k\omega_0)$$

and check by taking Inverse Fourier Transform:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0) e^{j\omega t} d\omega = e^{jk\omega_0 t}$$

by the sifting property.

Therefore:

$$e^{jk\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - k\omega_0)$$

and by linearity of the Fourier Transform,

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \longleftrightarrow \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_0)$$

So a periodic signal has a Fourier Transform that is an infinite impulse train at discrete frequencies $k\omega_0$ with weights of $2\pi C_k$ (many or most of the weights are usually 0).

Ex.

Find the Fourier Transform of $\cos\omega_0 t$.

Pulsed Cosine

Ex. Find the Fourier Transform of the pulsed cosine:
$x(t) = rect(\frac{t}{T})\cos(\omega_0 t)$ where again

$$rect(\frac{t}{T}) = \begin{cases} 1, & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

You can do it directly and also using the multiplication property.

5.4 Application of the Fourier Transform

Frequency Response of Linear Systems

We will see that we can avoid doing convolutions by using Fourier Transforms, or equivalently, we can solve differential equations algebraically by taking Fourier Transforms.

We saw in Chapter 3 that LTI systems have I/O relationship described by convolution:

$$y(t) = x(t) * h(t)$$

Due to the convolution property of Fourier Transforms

$$Y(\omega) = X(\omega)H(\omega)$$

where $H(\omega)$ is frequency response or transfer function of the LTI system.

An example of an $H(\omega)$ is a FILTER that *passes* frequencies of the input signal that are in a certain range and *attenuates* all other frequencies. You will see examples of filtering in EE341.

5.5 Energy and Power Density Spectra

We will not cover this section of the textbook.

Chapter 6 Applications of the Fourier Transform

We will see a number of topics in this chapter covered in your laboratories. In particular, we will investigate different filters in your labs. We will cover specific examples in class and add more, time permitting.

6.1 Ideal Filters

Given that an LTI system has the relationship $y(t) = x(t) * h(t)$ where $x(t)$ is the input, $y(t)$ is the output, and $h(t)$ is the impulse response of the system, we know that $Y(\omega) = X(\omega)H(\omega)$.

$H(\omega)$ can be an ideal filter to eliminate unwanted parts of the frequency spectrum. There are a number of different types of filters including ideal low pass, high pass, band pass, and band stop. The filters have Fourier Transforms of the form $rect(\frac{\omega}{2\omega_c})$, where ω_c is the cutoff frequency. The corresponding time domain functions are sincs.

Ex. Given two functions $x_1(t) = \cos(500\pi t)$ and $x_2(t) = \cos(1000\pi t)$, form a new signal $x_3(t) = x_1(t)x_2(t)$.

Draw the frequency response of a low pass filter that passes the low frequency component of the signal and blocks the high frequency component of the signal.

6.2 Real Filters

This section discusses real filters that can be implemented with Matlab. We will not cover most of it here but you will see some of these filters in EE341 and use them to filter simple data sequences.

6.3 Bandwidth Relationships

The bandwidth of a signal is the range of frequency values over which it is nonzero. The textbook discusses several definitions of bandwidth. The most important definitions are the

- Absolute bandwidth – the range over which the frequency response has a nonzero value
- 3-dB or half-power bandwidth – the range over which the frequency response has value no less than $\frac{1}{\sqrt{2}}$ the maximum value of the spectrum. The term 3 dB comes from

$$20 \log_{10}(\frac{1}{\sqrt{2}}) = -3 \text{ dB}$$

  This is widely used in the EE community. It comes from the fact that if the voltage or current is divided by $\sqrt{2}$, the power delivered to a load by that signal is halved.

An important fact is that the time duration of a signal and its bandwidth in frequency are inversely related.

- An impulse has zero time duration and infinite bandwidth
- A signal that changes suddenly in time or space (such as an edge in an image) will have a high frequency component
- $\boxed{\text{Ex.}}$ Given a time signal $x(t) = rect(\frac{t}{\tau})$, plot its spectrum. Notice that as τ increases, the duration of the signal in time increases and the bandwidth decreases.

6.4 Sampling Continuous-Time Signals

Sampling a continuous time signal is used, for example, in A/D Conversion.

Sample a continuous time signal for CDs, computers, etc.

Define the continuous time pulse train as:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$x(t)$ is the continuous time signal we wish to sample

Let $x_s(t) = x(t)p(t)$ be the sampled signal (we get the sampled signal by multiplying it in time by an impulse train). Then,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Let $p(t)$ have a Fourier Transform $P(\omega)$.
Then,

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

by the multiplication property. $X(\omega)$ is the Fourier Transform of $x(t)$ and $X_s(\omega)$ is the Fourier Transform of $x_s(t)$.
Now find $P(\omega)$:

$$P(\omega) = \mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)]$$

Use the Fourier Transform of periodic signals since the pulse train is a periodic signal

$$\mathcal{F}[\sum_k C_k e^{jk\omega_0 t}] = \sum_k 2\pi C_k \delta(\omega - k\omega_0)$$

where C_k are the Fourier Series coefficients.
Find C_k for the periodic impulse train,

$$\begin{aligned} C_k &= \frac{1}{T}\int_T p(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_T \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T} = C_k \text{ for all } k \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}[\sum_{k=-\infty}^{\infty} \delta(t - kT)] &= \sum_k \frac{2\pi}{T}\delta(\omega - k\omega_0) \\ &= \sum_k \omega_0 \delta(\omega - k\omega_0) \end{aligned}$$

Thus, a impulse train in time has a Fourier Transform that is a impulse train in frequency.

The spacing between impulses in time is T, and the spacing between impulses in frequency is $\frac{2\pi}{T}$.

So increasing the spacing in time decreases the spacing in frequency and vice versa. This is an important result!

Back to $X_s(\omega)$

Let $\omega_s = \frac{2\pi}{T_s}$ be the <u>sampling frequency</u>, which corresponds to a sampling period of T_s, which we have been using. Therefore

$$P(\omega) = \sum_k \omega_s \delta(\omega - k\omega_s) = \sum_k \frac{2\pi}{T_s}\delta(\omega - k\omega_s)$$

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi}X(\omega) * P(\omega) = \frac{1}{2\pi}X(\omega) * [\frac{2\pi}{T_s}\sum_k \delta(\omega - k\omega_s)] \\ &= \frac{1}{T_s}\sum_k X(\omega - k\omega_s) \end{aligned}$$

or we get replicated, scaled version of $X(\omega)$.

ω_b is for "bandwidth"

Now, what if $\omega_s - \omega_b < \omega_b$?

We would get overlap of the islands or "aliasing." Therefore, we need $\omega_s - \omega_b > \omega_b$ or $\omega_s > 2\omega_b$ to avoid aliasing.

Sampling Theorem says we need $\omega_s > 2\omega_b$ to recover $x(t)$ from its samples– in other words, we need to sample at least twice the highest frequency to avoid aliasing. Usually, we choose a sampling rate a bit higher than twice the highest frequency since filters are not ideal.

We hear music up to $20kHz$ and CD sampling rate is $44.1kHz$.

A dog would need a higher quality CD since they hear higher frequencies.

Recover $x(t)$ from its sampled version $x_s(t)$ by using a LPF to recover the center island.

Ex. Given a signal $x(t)$ with Fourier Transform $X(\omega)$ with cutoff frequency ω_c as shown:

you are given three different pulse trains with periods $T_1 = \frac{\pi}{\omega_c}$, $T_2 = \frac{\pi}{2\omega_c}$, and $T_3 = \frac{2\pi}{\omega_c}$, draw the sampled spectrum in each case. Which case experiences aliasing?

Ex. The inverse Fourier Transform of the signal in the previous example is $x(t) = \frac{\omega_c}{2\pi} sinc^2(\frac{\omega_c t}{2})$. Draw the sampled signals using the sampling trains of the previous example ($T_1 = \frac{\pi}{\omega_c}$, $T_2 = \frac{\pi}{2\omega_c}$, and $T_3 = \frac{2\pi}{\omega_c}$). Notice how aliasing looks in the time domain.

6.5 Reconstruction of Signals from Sample Data

We will not cover this section here.

6.6 Sinusoidal Amplitude Modulation

Modulation is multiplying a signal in time by sinusoids to shift the signal to a desired frequency band. This is done, for example, by radio (television) to assign different parts of the frequency spectrum to different radio (television) stations.

Ex. AM radio – Double-sideband, suppressed carrier, amplitude modulation

Let $x(t)$ be a music signal. Form and transmit

$$y(t) = [x(t) + B]\cos(\omega_c t + \phi)$$

Let's look at its Fourier Transform.

$$\begin{aligned} \omega_c &= \text{carrier frequency of radio station, Ex. 770kHz} \\ B &= \text{carrier} \end{aligned}$$

Let $B = 0 \rightarrow$ suppressed carrier.
Let $\phi = 0$ for simplification.
Then

$$y(t) = x(t)\cos\omega_c t = \frac{1}{2}x(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right]$$

$Y(\omega) = ?$,

$$Y(\omega) = \frac{1}{2}X(\omega - \omega_c) + \frac{1}{2}X(\omega + \omega_c)$$

Your radio demodulates the signal,

$$\begin{aligned}
z(t) &= 2y(t)\cos\omega_c t = 2x(t)\cos^2\omega_c t \\
&= x(t)[1+\cos 2\omega_c t] = x(t)[1+\frac{1}{2}e^{j\omega_c 2t}+\frac{1}{2}e^{-2j\omega_c t}] \\
Z(\omega) &= X(\omega)+\frac{1}{2}X(\omega-2\omega_c)+\frac{1}{2}X(\omega+2\omega_c)
\end{aligned}$$

To recover $x(t)$, filter $z(t)$ with a low-pass filter, i.e. multiply by a rect function in frequency–this corresponds to convolving with a sinc function in time.

Chapter 7 – The Laplace Transform

1. Using Laplace Transform, differential equations can be solved algebraically.

2. Can use pole/zero diagrams to determine frequency response and stability of a system.

3. Is used for analog circuit design.

4. Can transform more signals than by the Fourier Transform since the Fourier Transform is a special case of the Laplace Transform.

5. Used in Control Theory and Robotics

7.1 Definitions of Laplace Transform

Bilateral Laplace Transform

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Inverse Bilateral Laplace Transform

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

But we will see more convenient ways to take the inverse transform than using contour integration. When taking the inverse transform, the value of c for the contour integral must be in the region where the integral exists.

1. $s = \sigma + j\omega$
2. Converges for values of s in addition to $j\omega$ (which was Fourier Transform because $\sigma = 0$)

If we define $x(t)$ to be 0 for $t < 0$, that gives us the unilateral Laplace transform

$$\mathcal{L}[x(t)] = X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

An important difference between bilateral and unilateral Laplace Transforms is that you need to pay more attention to the region of convergence (ROC) for the bilateral case.

Taking the Laplace Transform is clearly a linear operation:

$$\mathcal{L}[ax_1(t) + bx_2(t)] = aX_1(s) + bX_2(s)$$

7.2 Examples

Ex. 1 Find $\mathcal{L}[u(t)]$

The Fourier Transform of $u(t)$ is $\pi\delta(\omega) + \frac{1}{j\omega}$ which requires an impulse.

Ex. Find the Laplace Transform for

$$x_1(t) = e^{-t}u(t)$$

The general Laplace Transform for an exponential function is:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, Re(s+a) > 0$$

7.3 Laplace Transforms of Functions

Ex. Find the Laplace Transform of $\delta(t - t_0)$ where $t_0 \geq 0$.

Ex. Find the Laplace Transform of $\sin(bt)u(t)$ and $e^{-at}\sin(bt)u(t)$.

You can derive all the Laplace Transforms in Table 7.2 on page 301 for more practice.

7.4 Laplace Transform Properties

As we saw from the Fourier Transform, there are a number of properties that can simplify taking Laplace Transforms. I'll cover a few properties here and you can read about the rest in the textbook.

Real Time Shifting

$$x(t) \leftrightarrow X(s)$$

$$x(t-t_0) \leftrightarrow e^{-t_0 s}X(s)$$

Derive this:

Plugging in the time-shifted version of the function into the Laplace Transform definition, we get:

$$\int_{t=-\infty}^{\infty} x(t-t_0)e^{-st}dt$$

Let $\tau = t - t_0$, we get:

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}d\tau$$

$$= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau$$

$$= e^{-st_0}X(s)$$

Ex. Find the Laplace Transform of $\sin b(t-2)u(t-2)$

Differentiation

$$x(t) \leftrightarrow X(s)$$

$$x'(t) \leftrightarrow sX(s)$$

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

Take the derivative of both sides of this equation with respect to t:

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} sX(s)e^{st}ds$$

Integration

$$x(t) \leftrightarrow X(s)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s)$$

7.5 Additional Properties

Multiplication by t

$$x(t) \leftrightarrow X(s)$$

$$tx(t) \leftrightarrow -\frac{dX(s)}{ds}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Take the derivative of both sides of this equation with respect to s:

$$\frac{d}{ds}X(s) = -\int_{-\infty}^{\infty} tx(t)e^{-st}dt$$

Independent-Variable Transformation

$$x(t) \leftrightarrow X(s)$$

$$x(at-b) \leftrightarrow \frac{e^{\frac{-sb}{a}}}{a}X(\frac{s}{a})$$

7.6 Response of LTI Systems

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $h(t)$ is an impulse response, is called the system function or transfer function and it completely characterizes the input/output relationship of an LTI system. We can use it to determine time responses of LTI systems.

Initial Conditions

We will not cover this topic.

Transfer Functions

Differential equations for systems of the form:

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$$

can be solved with the Laplace Transform:

$$Y(s)\sum_{k=0}^{n} a_k s^k = X(s)\sum_{k=0}^{m} b_k s^k$$

Or define the transfer function $H(s)$ as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}$$

Added Material on Inverse Laplace Transforms (Partial Fraction Expansion)

$$F(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}, \quad m < n.$$

Partial-Fraction Expansion:

$$\begin{aligned} F(s) &= \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n} \end{aligned}$$

where

$$k_j = (s-p_j)F(s)|_{s=p_j}$$

Thus

$$f(t) = \sum_{j=1}^{n} k_j e^{p_j t} u(t).$$

Ex.

$$H(s) = \frac{3s+1}{s^2+6s+5}, \quad x(t) = e^{-3t}u(t).$$

Ex. Find the transfer function $H(s)$ for the differential equation.

$$y'(t) + 2y(t) = 3x'(t).$$

Ex. Now let the input to the system be $x(t) = 5u(t)$. Find $y(t)$.

Convolution

As we saw for the Fourier Transform

$$x(t) * h(t) \leftrightarrow X(s)H(s)$$

This is useful for studying LTI systems.

We can completely characterize an LTI system from:

1. The system differential equation
2. The system transfer function $H(s)$
3. The system impulse response $h(t)$

Ex. Find the step response $s(t)$ to

$$h(t) = e^{-t}u(t)$$

Hint: $u(t) \leftrightarrow 1/s$

Ex.

Find the output of an LTI system with $h(t) = e^{bt}u(t)$ to an input $x(t) = e^{at}u(t)$ where $a \neq b$.

Transforms with Complex Poles

If an impulse response is real, the poles and zeros occur in complex conjugate pairs. This leads to sinusoidal terms in the steady state response.

Functions with Repeated Poles

We will not cover this section

7.7 LTI System Characteristics

Stability

We saw that a condition for bounded-input bounded-output stability was:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Let's look at stability from a system function standpoint. Given the Laplace Transform $H(s)$ for an impulse response $h(t)$, we have:

$$H(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}$$

The impulse response is:

$$h(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \cdots + k_n e^{p_n t} u(t).$$

What happens to $h(t)$ as $t \to \infty$? For a system to be stable, it must not become unbounded as $t \to \infty$.

If $Re\{p_i\} < 0$, $\forall i$, then $h(t)$ decays to 0 as $t \to \infty$ and system is stable (just evaluate $\int_{-\infty}^{\infty} |h(t)| dt$).

Therefore all poles of $H(s)$ in the left half plane of the s-plane if and only if the system is BIBO stable.

When you study CONTROL THEORY, you will hear lots more about this. You can build systems to steer the poles into the left half plane and thus stabilize the system.

Ex. FEEDBACK

a) you are given an impulse response

$$h(t) = e^t u(t).$$

Is the system stable?

b) You now hook up h(t) into a"Feedback" system as shown. Find the new impulse response or transfer function. Is this new system stable?

If you take EE446 you'll see this again!

Invertibility

You can find the inverse of a system using Laplace Transforms. This is because:

$$h(t) * h_I(t) = \delta(t)$$

Take the Laplace Transform of both sides:

$$H(s)H_I(s) = 1$$

Therefore, the Laplace Transform of the inverse system is simply

$$H_I(s) = \frac{1}{H(s)}$$

Ex. Find the inverse to

$$H(s) = \frac{s+a}{s+b}, Re\{s\} > -b \text{ and } Re\{s\} > -a$$

7.8 Bilateral Laplace Transform

We will see that for the Bilateral Laplace Transform we must specify Region of Convergence (ROC) because multiple time signals have the same Laplace Transform

$$X_b(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Must specify where integral converges.

$\boxed{\text{Ex.}}$ Find $\mathcal{L}[e^{4t}u(t)]$

(The Fourier Transform does not exist in this case).

Ex. Find $\mathcal{L}[-e^{4t}u(-t)]$ and plot its ROC. Does the Fourier Transform exist?

Therefore, MUST SPECIFY ROC!

Also, the ROC of the Laplace Transform of the sum of multiple time functions is the INTERSECTION of the individuals ROCs.

Region of Convergence

Ex. Find $\mathcal{L}[e^{-3t}u(t) - e^{-t}u(-t)]$ and plot its ROC.

Ex. Find $\mathcal{L}[e^{-t}u(t) - e^{-3t}u(-t)]$ and plot its ROC.

Bilateral Transform from Unilateral Tables

We will not cover this topic

Inverse Bilateral Laplace Transform

We'll use partial fraction expansion and ignore the case of multiple poles: Write

$$\frac{B(s)}{A(s)} = b_N + \sum_{k=1}^{N} \frac{r_k}{s+s_k}$$

s_k = poles, r_k = residues, b_n is nonzero if the order of numerator is greater than the order of denominator.

Now, we can just use transforms we already know.

Ex.1 Invert

$$\frac{2s+6}{s^2+6s+8}, \quad Re\{s\} > -2$$

Write as:

$$\frac{2s+6}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}$$

Solve for A and B: $2s+6 = A(s+4) + B(s+2)$.
YOU FINISH!

Ex. 2 Find the Inverse Laplace Transform of

$$\frac{4s+2}{s^2+3s+2}, \quad Re\{s\} < -2$$

Ex. 3 Find the Inverse Laplace Transform of

$$\frac{2s+1}{s^2+s}, \quad -1 < Re\{s\} < 0$$

You can always check your results by taking the transform and comparing.

Laplace Transform Notes

Bilateral vs. unilateral Laplace Transform

The *bilateral* (or, two-sided) Laplace transform and its inverse are defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}dt$$

where c is any real value in the region of convergence of $X(s)$. The *unilateral* (or, one-sided) Laplace transform is very similar except that the lower time limit is zero:

$$X_u(s) = \int_{0}^{\infty} x(t)e^{-st}dt,$$

assuming that anything that happened before time $t = 0$ is not of interest. If the signal $x(t) = 0$ for $t < 0$ then these two transforms are identical. If you want to characterize a signal or system behavior for all time, then the bilateral transform is what you should use. If you want to characterize a system response when you have some initial condition, then the unilateral tranform is what you should use. When the system is initially at rest, both give the same answer.

The initial and final value theorems are useful when $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher order singularities at $t = 0$

$$x(0+) = \lim_{s\to\infty} sX(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Table 1: Laplace transform properties for bilateral vs. unilateral case. The new ROC R' is given in the ROC column as a function of the ROC of associated signals.

	Bilateral			Unilateral			
	$x(t)$	$\leftrightarrow$	$X(s)$	$x(t)$	$\leftrightarrow$	$X_u(s)$	
linearity	$ax_1(t) + bx_2(t)$	$\leftrightarrow$	$aX_1(s) + bX_2(s)$	$ax_1(t) + bx_2(t)$	$\leftrightarrow$	$aX_{1u}(s) + bX_{2u}(s)$	
integral	$\int_{-\infty}^{t} x(t)dt$	$\leftrightarrow$	$\frac{X(s)}{s}$	$\int_0^t x(t)dt$	$\leftrightarrow$	$\frac{X_u(s)}{s}$	R'
derivative	$\frac{dx(t)}{dt}$	$\leftrightarrow$	$sX(s)$	$\frac{dx(t)}{dt}$	$\leftrightarrow$	$sX_u(s) - x(0+)$	
multiply by t	$tx(t)$	$\leftrightarrow$	- $\frac{dX(s)}{ds}$	$tx(t)$	$\leftrightarrow$	- $\frac{dX_u(s)}{ds}$	
convolution	$x(t) * h(t)$	$\leftrightarrow$	$X(s)H(s)$	$x(t) * h(t)$	$\leftrightarrow$	$X_u(s)H_u(s)$	
time shift	$x(t - b)$	$\leftrightarrow$	$e^{-sb}X(s)$	$x(t - b);\ b > 0$	$\leftrightarrow$	$e^{-sb}X_u(s)$	
s-domain shift	$e^{s_0 t}x(t)$	$\leftrightarrow$	$X(s - s_0)$	$e^{s_0 t}x(t)$	$\leftrightarrow$	$X_u(s - s_0)$	
time scaling	$x(at)$	$\leftrightarrow$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	$x(at);\ a > 0$	$\leftrightarrow$	$\frac{1}{a}X_u(\frac{s}{a})$	
time flip	$x(-t)$	$\leftrightarrow$	$X(-s)$		n/a		
conjugate	$x^*(t)$	$\leftrightarrow$	$X^*(s^*)$	$x^*(t)$	$\leftrightarrow$	$X_u^*(s^*)$	

Table 2: Laplace transform pairs. (These are for the bilateral case, but the unilateral transforms are the same for the cases where $x(t) = 0$ when $t < 0$.)

$x(t)$	X(s)	ROC	comments
$\delta(t)$	1	All s	
$u(t)$	$\frac{1}{s}$	$Re\{s\} > 0$	
$tu(t)$	$\frac{1}{s^2}$	$Re\{s\} > 0$	
$-tu(-t)$	$\frac{1}{s^2}$	$Re\{s\} < 0$	
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$Re\{s\} > 0$	
$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$Re\{s\} < 0$	
$e^{-at}u(t)$	$\frac{1}{a+s}$	$Re\{s\} > -Re\{a\}$	
$-e^{-at}u(-t)$	$\frac{1}{a+s}$	$Re\{s\} < -Re\{a\}$	
$e^{-a\|t\|}$	$\frac{1}{a^2-s^2}$	$-a < Re\{s\} < a$	real $a > 0$
$t^n e^{-at}u(t)$	$\frac{n!}{(a+s)^{n+1}}$	$Re\{s\} > -Re\{a\}$	
$-t^n e^{-at}u(-t)$	$\frac{n!}{(a+s)^{n+1}}$	$Re\{s\} < -Re\{a\}$	
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$	$Re\{s\} > 0$	
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$	$Re\{s\} > 0$	
$\cos(bt+\phi)u(t)$	$\frac{s\cos\phi - b\sin\phi}{s^2+b^2}$	$Re\{s\} > 0$	
$t\sin(bt)u(t)$	$\frac{2bs}{(s^2+b^2)^2}$	$Re\{s\} > 0$	
$t\cos(bt)u(t)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$Re\{s\} > 0$	
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$	$Re\{s\} > -a$	real a
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$Re\{s\} > -a$	real a

What can you learn from a pole zero plot?

1. From the "shape" of the ROC, you can learn direction of the time signal, or determine if a system is *not* causal.

Property	Time Signal	System Property
left-sided	left-sided	not causal
right-sided	right-sided	possibly causal
strip	two-sided	not causal
whole plane	finite duration	possibly causal

2. If the $j\omega$-axis is included in the ROC, then the time signal is absolutely integrable (i.e. the Fourier transform exists). If the pole-zero plot represents a system response, then the system is stable.

3. The components of a time signal, or the modes of the system (terms in the natural response), have an oscillating component if any poles are off the real axis and/or a decaying (or growing) component if any poles are off the imaginary axis. The poles are at -0.5 (decaying), $\pm j5$ (oscillating), and $-0.5 \pm j5$ (both), respectively in examples below, all with a right-sided ROC.

4. A time signal or a system is *real* if the poles and zeroes come in complex conjugate pairs, i.e. $a \pm j\omega_0$. (A system being real means that the impulse response is real, so that if the input time signal is real, then the output will be real.)

5. Putting together (1) and (2): for an LTI system to be both stable and causal, all its poles must be in the left-half plane.